Uranium-238 (^{238}U or U-238) spontaneously decays to thorium-234 (^{234}Th or Th-234) by emitting an alpha particle, 2 protons and 2 neutrons. The decay rate is proportional to the amount of U-238 present at any one time,

$$\frac{dN}{dt} = -kN$$

where N is the number of atoms (or proportional measure, such as moles) and k is the decay rate constant, often written also as λ. The constant is related to the half-life τ as $k = ln2/\tau$, with ln2 being the natural logarithm of 2 with the value 0.693… The solution of the simple single-decay equation is the negative or declining exponential,

$$N = N_0 e^{-kt}$$

For U-238 the half life is long, about the age of the Earth, 4.5 billion years.

The Th-234 itself decays into protactinium-234 or Pa-234 by emitting a beta particle or electron. The half life of Th-234 is dramatically shorter, at 24.1 days. In turn, Pa-234 decays into uranium-234 by emitting another beta particle, with a very short half life of 6.7 hours. U-234 decays with a long half life of 245,00 years. There are further decays in a long chain that ends with the stable element lead-206.

A mathematical challenge is solving for the amounts of U-238, Th-234, Pa-234, Th-234, and so on at any given time, starting with known initial amounts of each. We might consider these first four nuclides. An ancient ore sample might contain amounts of each nuclide that have essentially the same activity or disintegrations per unit time. An arbitrary sample might have a quite different balance among the nuclides because they were washed out at differing rates. So, the task is this: knowing the amount of each of the four nuclides at a starting time that we'll call 0, calculate the amounts of each at a later time, t.

We have to solve the series of differential equations, one for each decay process. Let's denote the nuclides with single letters, A for U-238, B for Th-234, C for Pa-234, and D for U-234. The decay constants are respectively k_A, k_B, k_C, k_D. The equations are deceptively simple,

$$\frac{dA}{dt} = -k_A A$$

$$\frac{dB}{dt} = -k_B B$$

$$\frac{dC}{dt} = -k_C C$$

$$\frac{dD}{dt} = -k_D D$$

Let's consider times much shorter than the half-life of U-238, so that A is essentially constant. We'll see many interesting phenomena even within this scope. We'll then set A=1 in some units, such as 1 mole, Avogadro's number of U-238 atoms.

At this point we have divergent options for solving the equations. An obvious option is numerical integration. Set time steps, which may vary adaptively, and just solve for A, B, C, and D at each time step. There are sophisticated methods that minimize the computational time. There is, however, the constraint that, to get to the amounts of A, B, C, and D at any given time, the integration must be run in

full, or at least in segments, each of which is a full calculation. There's no way to get an immediate answer. A much bigger problem is that the differential equations taken together form what is called a set of stiff differential equations. The rates of decay in the U-238 decay chain vary by 21 orders of magnitude, deriving from half-lives varying from 45 billion years to 130 microseconds. Naïve numerical integration has to keep the time step at the shortest among stages in the decay chain, which is completely impractical. Mathematically interesting transformations of the coupled decay equations allow accurate and practical calculations without numerical integration; they amount to very high-level reformulations of the algebra used in this discussion.