Some more about the exponential function, e^x

In the derivation of the Poisson distribution, the exponential function came up naturally, via one of its several definitions, as the limit as the number of intervals gets indefinitely large (*N* → ∞ , i.e., infinity) of this function:

$$\lim_{N\to\infty}\left(1+\frac{x}{N}\right)^N = e^x$$

A practical way to look at the exponential is in describing compound interest. Consider a gain in value (interest) at a rate *a* (per day, per month, per year, or whatever). How fast does some initial value, $V_{0,}$ increase? If we do simple interest, adding a fraction *at* after a time *t* has elapsed, then, if the interest keeps accumulating for a total time T, the final value is

$$V = V_0(1+aT)$$

Now let us add interest at every time interval in an amount equal to the fraction of the new, current value. Let's divide the final time into *N* small intervals, *T/N*. Then, the value rises as

$$V_1 = V_0\left(1+a\frac{T}{N}\right)$$

$$V_2 = V_1\left(1+a\frac{T}{N}\right) = V_0\left(1+a\frac{T}{N}\right)\left(1+a\frac{T}{N}\right)$$

$$V_3 = V_2\left(1+a\frac{T}{N}\right) = V_0\left(1+a\frac{T}{N}\right)\left(1+a\frac{T}{N}\right)\left(1+a\frac{T}{N}\right)$$

......

$$V_N = V_0\left(1+\frac{aT}{N}\right)^N \to V_0 e^{aT}$$

Let's look at how we can calculate e^{aT} (pronounced as "e to the aT"). Let's start by writing out the values of V_3 and V_4:

$$V_3 = V_0\left[1+3\left(a\frac{T}{N}\right)+3\left(a\frac{T}{N}\right)^2+\left(a\frac{T}{N}\right)^3\right]$$

$$V4 = V_0\left[1+3\left(a\frac{T}{N}\right)+4\left(a\frac{T}{N}\right)^2+3\left(a\frac{T}{N}\right)^3+\left(a\frac{T}{N}\right)^4\right]$$

We can keep going, and you can convince yourself that the final expression is

$$V_N = V_0\left[1+N\left(a\frac{T}{N}\right)+\frac{N(N-1)}{2}\left(a\frac{T}{N}\right)^2+\frac{N(N-1)(N-2)}{2*3}\left(a\frac{T}{N}\right)^3+...\right] \to V_0 e^{aT}$$

When N is very large, N(N-1) looks indefinitely close to N^2, N(N-1)(N-2) looks like N^3, and so on. This give us

$$e^{aT} = 1+(aT)+\frac{1}{2}(aT)^2+\frac{1}{6}(aT)^3+\frac{1}{24}(aT)^4+...$$

All the terms after the first two are in excess of those with simple interest. This equation is the second definition of the exponential. It also gives us a way to compute the exponential as an infinite series, which always converges to a finite, final value. One special value of e^x is for *x=1*:

$$e = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + ... = 2.71828183...$$

This is one of the fundamental mathematical constants, along with π and several other common ones. If you want to impress people, memorize, say, 25 digits of π (lots of people try this) and then 25 digits of *e* (very few people know this).

Let's see how much different compound interest is. Let *a* be 5% = 0.05 per year. Let the total time be 1 year:

Simple interest: the value increases by a factor (1+0.05*1) = 1.05

Compound interest, compounded every tiny interval: the value increases by a factor

$$e^{0.05} = 1 + 0.05 + \frac{1}{2}(0.05)^2 + \frac{1}{6}(0.05)^3 + ... = 1.0513$$

This is bigger than simple interest by the sum of all the terms after +0.05. Sure, it's not much bigger, but now let's run this for 50 years, as if you held stocks that gained 5% per year for that long. Simple interest would give you an increase by a factor 1+50*0.05=3.5. Compound interest would give you an increase by a factor $e^{50*0.05} = e^{2.5}$=12.18!

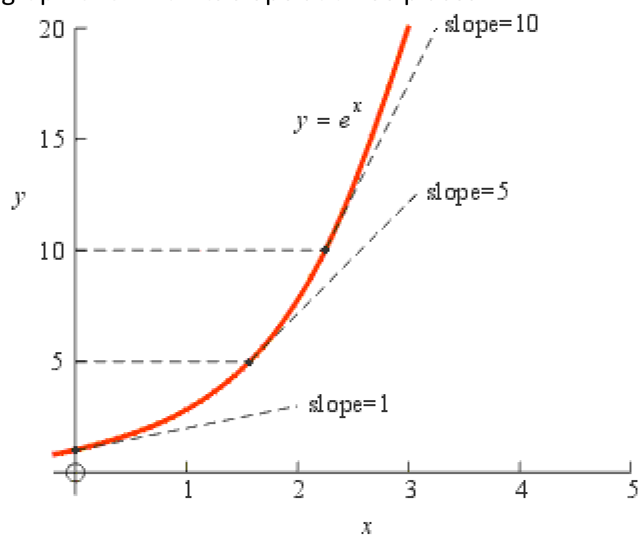
The positive exponential increases faster at long times (large values of its "argument") than any polynomial function of a finite number of terms, largely because it has an infinite number of terms.

One can use the infinite series shown above in order to compute the exponential e^x for any value of its argument, *x*. There are more efficient ways, used particularly in computers, to do this computation. There's no need to go into detail at this time.

A third way to define the exponential, e^x, is that its rate of increase at any value, *x*, is e^x itself. You'll run into differential equations later, which will state that e^x is (defined as, a fourth time) as the solution for the function *y* of the differential equation

$$\frac{dy}{dx} = y$$

Let's take this third definition, and see if the definition as the power series satisfies the third definition. We need to compute the derivative as the sum of the derivatives of all the individual terms. That requires that we know the concept of a derivative. It is the slope of the graph of a function, simply. Here's a graph of e^x with its slope at three places[1]:



That means the rate of change over the smallest of intervals. You'll run into this in calculus as the derivative *f'(x)* of a function *f(x)* at being defined as the difference in the value of that function at *x+h* and at *x*, divided by that increment *h* as *h* becomes infinitely small. That is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This is also commonly written as $\frac{df(x)}{dx}$

There are some tricky parts for special kinds of functions, which we'll ignore. Also, *h* can be positive or negative and the result has to be the same if the function is said to be differentiable.

We can work out the derivatives of powers of *x*:

For $f(x)=x^0=1$, a constant, $f(1)-f(1) =0$; the derivative of a constant is zero

For $f(x)=x^1=x$, we have $f(x+h)-f(x) =x+h - x = h$ and, thus, $f'(x) = h/h = 1$

For $f(x)=x^2$, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h}$$
$$= \lim_{h \to 0} (2x + h) = 2x$$

For $f(x)=x^3$, we have

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h \to 0} (3x^2 + 3xh + h^2) = 3x^2$$

In general, for $f(x)=x^n$, we have $f'(x)=nx^{n-1}$

We can then differentiate the power series:

$$\frac{de^x}{dx} = \frac{d}{dx}(1) + \frac{d}{dx}(x) + \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}\left(\frac{x^3}{6}\right) + \frac{d}{dx}\left(\frac{x^4}{24}\right) + \ldots$$
$$= 0 + 1 + \frac{2x}{2} + \frac{3x^2}{6} + \frac{4x3}{24} + \ldots$$
$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$
$$= e^x$$

So, yes, it works.

How about the negative exponential?

This function, e^{-x}, comes up naturally in decay processes, such as of radioactive elements. You've heard of half-lives, often written as $t_{1/2}$. In a time $t_{1/2}$, half of the element decays. In the next interval of length $t_{1/2}$, half of the remaining half decays, leaving ¼ of the original amount. In general, after an arbitrary time *T*, the amount left is $e^{-T/t1/2}$, better written out as $exp(-T/t_{1/2})$.

To derive this, we use the same kind of argument as for the positive exponential, but with a negative sign. In a tiny interval *t*, the decrease in value is from V_0 to $V_0(1-kt)$, writing *k* for a decay constant (we'll find that it's inversely related to the half life, as $k=0.693/t_{1/2}$. This makes sense – a shorter half life is a faster decay, a larger decay constant).

Again breaking up a finite time interval *T* into a very large number, *N*, of tiny intervals, we have

$$V(T) = V_0\left((1-k\frac{T}{N} \right)^N$$

In the limit of infinite *N*, this looks like V_0 times an exponential, but with the argument being negative, -*kT*.

Let's now look at how *k* is related to $t_{1/2}$. We have to satisfy the relation

$$\exp(-kt_{1/2}) = \frac{1}{2}$$

We have to introduce the companion idea of logarithms, which are the inverse functions for exponentials. Basically, a number *y* is the natural logarithm of x if $x=e^y$. This is similar to the idea of logarithms to the base 10, where, for example, 3 is the logarithm to the base 10 of the number 1,000 ($\log_{10}(1000)$ =3).

We take the natural log of both sides of the display equation above:

$$-kt_{1/2} = \ln\left(\frac{1}{2}\right) = -\ln 2 \rightarrow kt_{1/2} = \ln 2 = 0693...$$

$$\rightarrow k = \frac{\ln 2}{t_{1/2}}$$

In the above, I used the fact that the logarithm (natural, or base 10, or any base) of an inverse power of a number is the negative of that power. For example,

$$\ln\left(\frac{1}{1000}\right) = \ln\left(\frac{1}{10^3}\right) = \ln(10^{-3}) = -3$$

which is

$$-\ln(1000) = -(3)$$

A couple of examples of decay: first, How much of the original radioactive potassium-40 on Earth has already decayed, since the Earth formed 4.8 billion years ago? The half-life of ^{40}K is 1.25 billion years. We have

$$e^{-kt} = e^{-0.693t/t_{1/2}} = e^{-0.693x4.8/1.25}$$

$$= e^{-2.66} = 0.070$$

That is, only 7% is left, or 93% of it has decayed. That put a lot of heat into the Earth, which is still coming out at the surface, though the total of all the heat from radioactive decay, gravitational accretion, and other processes makes a heat flux of only 0.06 watts per square meter. Second, What if the vitamin C in your refrigerated orange juice gets oxidized away by ½ in a week. How much did you lose in the first day? We have *k=0.693/7 days=0.099* per day. In one day, the fraction left is

$$e^{-k*1day} = e^{-0.099} = 0.905$$

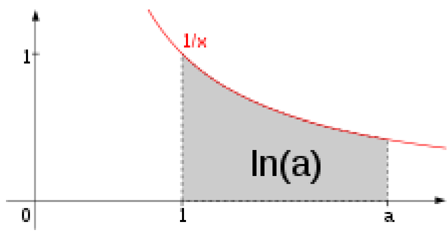
So, you lost 9.5%.

How are logarithms defined and calculated, anyhow?

The logarithm has a "simple" definition, as an integral

$$\ln(x) = \int_1^x \frac{dy}{y}$$

You'll run into integrals in calculus, too. They are basically the area under a curve as it is graphed out. For example[2], the natural log of *e*, ln(*e*), is 1:



That might not be obvious as the area, I admit.

The numerical value of *ln(x)* can be calculated by simple integration, though it's not the most efficient way in computers and the like. Still, here's a cute calculation. Let's calculate *ln(2)* by breaking up the ordinate between values 1 and 2 into 10 intervals. In each one, we have the width 0.1. In each interval, we'll approximate $1/y$ as a simple constant, $1/y_{mid}$, with y_{mid} as the value at the midpoint – that is, 1.05 for the first interval, 1.15 for the second interval, and so on. We get the approximate value

$$\ln(2) \approx 0.1^* \left(\frac{1}{1.05} + \frac{1}{1.15} + \frac{1}{1.25} + \frac{1}{1.35} + \ldots + \frac{1}{1.95} \right)$$
$$= 0.69284$$

The true value is 0.69314. Our simple method is accurate to within 0.05%. The method gets intractable for large values of *x*.

It should be clear that the *ln(x)* grows increasingly more slowly as *x* gets large. Just from the definition as the integral, the value added to *ln(x)* at large x gets small as 1/y gets small. This is to be expected. If *ln(x)* is the inverse of e^x, which grows very fast, then it should grow slowly.

What is not immediately apparent from their definitions is that *ln(x)* and e^x are exact inverses of each other. For example, if we use the power series expression for e^x, it's not clear that this is true:

$$x = e^{\ln(x)}$$
$$= 1 + \ln(x) + \frac{1}{2}[\ln(x)]^2 + \frac{1}{6}[\ln(x)]^3 + \ldots$$

especially if we write out *ln(x)* as the integral used earlier.

Probably the simplest way to grasp that these functions are inverses is to use the fact that the exponential is its own derivative,

$$\frac{de^x}{dx} = e^x$$

Let's use this in the definition of *ln(x)*:

$$\ln(e^x) = \int_1^{e^x} \frac{dy}{y}$$

We will rewrite the right-hand side into new variables. Let's write

$$y = e^z$$

$$\text{This also give us } \frac{dy}{dz} = e^z \quad \text{or } dy = e^z dz$$

The limits of integration get changed to new values; $e^z=1$ translates to z=0, so that the lower limit becomes 0. For the upper limit, $e^z = e^x$ has $z=x$, so the upper limit becomes x.

Now we have

$$\int_0^x \frac{dz\, e^z}{e^z} = \int_0^x dz = x - 0 = x$$

I used some further knowledge about integrals here, such as $\int_0^x dz = x - 0 = x$, which could take a little explaining, as could the legitimacy of changing variables in the integral and doing the corresponding changes in the limits of integration. I'll skip these here. In any case, the natural logarithm and the exponential are inverses of each other; $ln(e^x)=x$ and $e^{ln(x)}=x$.

One final note, not really in math but in subjective experience. Here is my interpretation of why the years seem to go by so fast as one gets older, compared to the interminable length of a year when one is young. Basically, the subjective experience of a year (or other time interval) is proportional to the *fraction* of new experiences in that time. The increment in new things in a time interval *dt* might be proportional to that interval, such as *k*dt*, with *k* as some rough constant. However, the cumulative experience to the real time in question, *t*, is proportional to that total time, like *k*t*. The increment in subjective time is *(k*dt)/(k*t) = dt/t*. Summing up subjective time is the same as doing the integral, so that subjective time *S(t)* looks like

$$S(t) = \int_1^t \frac{dt'}{t'} = \ln(t)$$

I conveniently set the lower limit to age 1 year, but it doesn't matter.

Let's compare subjective time up to age 6 to its values at age 8 (that was a long transition, wasn't it, from end of kindergarten to end of 2nd grade?!), age 10...and then age 20, age 30, age 40, age 50, and age 60:

t	S(t)	Gain in S(t)
6	1.79	
8	2.08	0.29
10	2.30	0.22
20	3.00	0.70
30	3.40	0.40
40	3.69	0.29
50	3.91	0.22
60	4.09	0.18

So, there's a much greater experience of the passage of time in the 2 years between ages 6 and 8 than in the 10 years between ages 50 and 60. Lesson: to enjoy the feeling of living long, do as many new things as possible each year as you age!

A couple of references, for the graphs taken from the Web:

[1} http://mathonweb.com/help_ebook/html/expoapps.htm. Accessed 27 March 2015.

[2] http://en.wikipedia.org/wiki/Natural_logarithm. Accessed 27 March 2015.