First, what it the equation for acceleration of the rocket in the boost phase, when the engine is burning propellant?

The basic equation for a body is Force = F = dp/dt = (d/dt)(mv)
We have to be careful – there are two "bodies" – the rocket body and the exhaust gas left behind.

At one instant, we have   A short time later, after burning an amount Δm of propellant:

v → v_{ex} →   v+Δv →

F→ [m]   [Δm]   F→ [m-Δm]   Here, F is any other set of forces, such as gravity + drag

p = mv   p+Δp = Δm v_{ex} + (m-Δm)(v+Δv)
= Δm v_{ex} + mv – Δm v + mΔv – higher-order terms
= (v-v_{ex})Δm + mΔv
= -uΔm + mΔv, where u = exhaust velocity relative to rocket

F = dp/dt = lim Δp/Δt = -u dm/dt + m dv/dt =- T + m dv/dt, where T = rocket thrust, u dm/dt
or
m dv/dt = T + F

So, in the boost phase of a rocket launched on or near the Earth, we have two forces, F: gravity, and drag. Let's consider a rocket moving vertically, so that we only have to concern ourselves with motion in one coordinate, the vertical coordinate, x.

The force of gravity is then simply –mg.

The drag force, F_{drag} is approximately ½ C_d A ρ v^2, where ρ is the mass density of air (about 1 kg m^{-3} in our conditions in Las Cruces, NM). This form is appropriate for bluff-body drag (pushing air out of the way, at subsonic velocities). For a pointed rocket nose, C_d is about ½, but it even decreases as velocity increases. There's also skin drag, which depends on rocket dimensions, etc., and pressure drag, which is a function of shape (streamlining, e.g.), but I'll approximate it as having the same dependence on velocity, just making C_d larger. Let's just call ½ A C_d an effective area, A'. There's an extensive, interesting discussion on drag forces of various kinds in Stephen Vogel's book, *Life in Moving Fluids*.

We now have dv/dt = (T/m – g) – (A' ρ /m) v^2 --> a – (A' ρ /m) v^2

Rearrange this to get terms in velocity, *v*, all on one side and *t* on the other: $\frac{dv}{a(1-\frac{A'\rho}{a\,m}v^2)} = dt$

With the change of variable $y = \sqrt{\frac{A'\rho}{am}}\,v$, we get the term in parentheses to be (1-y^2), which will help in the process of integrating over time, and overall we get $\frac{1}{a}\sqrt{\frac{am}{A'\rho}}\,\frac{dv}{(1-y^2)} = dt$

We can integrate this, using $\int \frac{dy}{(1-y^2)} = tanh^{-1}y,\ \ and\ we\ get\ \ t = \frac{1}{a}\sqrt{\frac{am}{A'\rho}}\,tanh^{-1}y$

Taking *tanh* of both sides, we then get $y = \tanh\left(\sqrt{\frac{A'\rho a}{m}}t\right)$, which gives $v = \sqrt{\frac{am}{A'\rho}}\tanh\left(\sqrt{\frac{A'\rho a}{m}}t\right)$

All the physical units check out nicely.

Now to plug in some numbers:

We used an Estes C6-5 engine, which has a thrust of 6 N for 1.67 s.

The mass of the empty rocket is 34 g and the engine weighs 25 g at the start. Let's use an average mass of 50 g total (the more accurate equation, accounting for the change of mass, might not have a closed-form solution; I haven't tried it yet).

Thus, T/m = 6 N/ 0.05 kg = 120 m s^{-1} and a = T/m –g = 110 m s^{-1}

The frontal area of the rocket is that of a circle 1" = 2.54 cm in diameter, or very nearly 5 cm^2 = $5x10^{-4}$ m^2. Let's take A'=this value.

The result is that the velocity at the end of boost is $v = \sqrt{\frac{110*0.05}{5x10^{-4}*1}}\tanh\left(\sqrt{\frac{5x10^{-4}*1*110}{0.05}} * 1.67\right)$

= 105 tanh(1.75) = 99 --> ca. 100 m s^{-1} (220 mph)

We can, of course, use the equation to estimate v at any time in the boost phase, as well as the acceleration, dv/dt, or the altitude at any time. Let's do the altitude:

Travel distance $x = \int dt\, v(t) = c_1 \int dt\, tanh\,(c_2 t) = \frac{c_1}{c_2}\int dy \tanh(y)\, , y = c_2 t$

Here, $c_1 = \sqrt{\frac{am}{A'\rho}}$ and $c_2 = \sqrt{\frac{A'\rho a}{m}}$

Using ∫*dy* tanh(*y*) = ∫*dy* sinh(*y*)/cosh(*y*) = ∫*d*(cosh(*y*)/cosh(*y*), we get $x = \frac{c_1}{c_2}\ln(\cosh(y))$

With $\frac{c_1}{c_2} = \frac{m}{A'\rho} = \frac{0.05}{5x10^{-4}*1} = 100\ (m\, s^{-1})$, and t = 1.67 s (the burnout time), and $c_2 = \sqrt{\frac{A'\rho a}{m}}$=1.05,

we get x=100 ln(cosh(1.75))=100 ln(2.96)=109 m

(The burnout time is simply the total impulse, 10 N s, divided by the average thrust, 6 N)

The velocity is a convex function of t, so that the average velocity is > ½ v_{final}, or $\bar{v}$ > 49.5 m s^{-1}, so that x > 49.5*1.67 = 83 m. This checks.

Now let's see how much higher the rocket goes as it continues upward in the coast phase, until gravity and drag slow its upward velocity to zero.

After the rocket motor cuts out, the rocket's equation of motion looks like

$\frac{dv}{dt} = -g - \frac{A'\rho}{m}v^2$, which gives $\frac{dv}{g\left(1+\frac{A'\rho}{mg}v^2\right)} = -dt$

A similar change of variables, $z = \sqrt{\frac{A'\rho}{mg}}v$ makes the stuff in parentheses become ($1+z^2$), and then we have dv = $\sqrt{\frac{mg}{A'\rho}}dz$ and the whole equation looks like

$$\sqrt{\frac{m}{A'\rho g}}\frac{dz}{(1+z^2)} = -dt$$

Now we need to integrate this on both sides. It can be shown that $\int \frac{dz}{(1+z^2)} = \tan^{-1}(z)$, so that we get

$$\sqrt{\frac{m}{A'\rho g}} \quad \tan^{-1}(z)| = -t$$

We can integrate from the beginning of coasting to the end (when v = 0, so that z = 0), to get

$$0 - \sqrt{\frac{m}{A'\rho g}}\tan^{-1}(z_{bf}) = -\, t_{cf}$$

where z_{bf} is the value at the end of the boost phase, $z_{bf} = \sqrt{\frac{A'\rho}{mg}}v_{bf}$, with v_{bf} = the velocity at the end of boost

Thus, the length of time from beginning to end of the coast phase can be found. Plugging in the numerical values for our rocket (A' = 5×10^{-4} m^2, ρ = 1 kg m^{-3}, g = 9.8 m s^{-1}, m = 0.05 kg, v_f = 99 m s^{-1}), we get

$$z_{bf} = \sqrt{\frac{5x10^{-4}}{0.05*9.8}}\,99 = 3.16 \quad \text{and} \quad t_f = \sqrt{10.2}\tan^{-1}3.16 = 4.04\ s$$

So, the engine "burns" for 1.67 s and the rocket coasts higher for another 4.04 s

We want to find out how much higher it goes, so we have to integrate the velocity over the coast time. At any time t within the coast phase (beginning with t=0), we have

$$v = \sqrt{\frac{mg}{A'\rho}}z \quad \text{and} \quad \tan^{-1}(z) - \tan^{-1}(z_{bf}) = -\sqrt{\frac{A'\rho g}{m}}t$$

with z_{bf} = z at the end of boost, or $z_{bf} = \sqrt{\frac{A'\rho}{mg}}v_f = 3.16$, as above

Let's denote $\tan^{-1}(z_{bf})$ = c, a constant over the coast phase. We then have, at any time *t* a velocity v that can be calculated from the corresponding value of y:

$$\tan^{-1}(z) = c - \sqrt{\frac{A'\rho g}{m}}t \qquad \text{or, taking } tan \text{ of both sides,} \qquad y = \tan\left(c - \sqrt{\frac{A'\rho g}{m}}t\right)$$

Let's denote $\sqrt{\frac{A'\rho g}{m}}$ as *s*, so that we write *z* = tan*(c-st)*.

Now we have

$$v = \sqrt{\frac{mg}{A'\rho}}\tan(c - st),$$ which we'll write more cleanly as $v = b\tan(c - st)$

We can now integrate over time. The gain in altitude, x, over the altitude at the end of boost, is

$$x = \int dt\, v = b\int dt \tan(c - st)$$

We'll change variables, to *t' = c-st,* to get

$$x = -\frac{b}{s}\int dt' \tan(t')$$

Now, $\int dt' \tan(t') = \int \frac{dt' \sin(t')}{\cos(t')} = \int \frac{-d(\cos(t')}{\cos(t')} = -\ln(\cos(t'))\,|$

Evaluating the integral at both limits, t' = c – 0 = c and t' = c-st$_f$ = 0 (as one can show), we have

$$x = -\frac{b}{s}[\ln(cos(c)) - \ln(cos(0)] = -\frac{b}{s}[\ln(\cos(c)) - \ln(\cos(0))] = -\frac{b}{s}\ln(\cos(c))$$

From the numerical values for our rocket, we have b/s = $\frac{m}{A'\rho} = 100\ m$ and c = $\tan^{-1}$(3.16) = 1.264. Thus, we get cos(c) = 0.302, ln(cos(c)) = -1.198, and, finally, x = 120 m.

The total altitude is then 109 + 120 = 229 m, or 722 ft. The rocket has gained more a bit more altitude during the coast phase than during the boost phase.