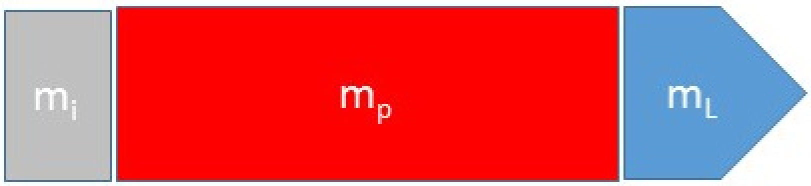
Let's look at how a real rocket has to allocate mass among propellant, payload....and the ever important infrastructure, meaning the combustion chamber, propellant tanks, fuselage, etc.

Let's denote the masses of infrastructure, propellant, and payload as m_i, m_p, and m_L, respectively:



Now, the rocket needs infrastructure in some proportion to the masses of propellant and of payload. Let's take this as a fixed fraction, α, of the sum of these two masses,

$$m_i = \alpha(m_p + m_L)$$

Let's see how the need for infrastructure affects the final velocity; recall that the final speed is a function of the natural log of the initial over final mass, with low final mass allowing high speed, but m_i is a mass that can't be shed.

The final velocity, relative the start at rest, is

$$v_f = v_{ex} \ln\left(\frac{m_0}{m_f}\right)$$

We'll assume that *all* of the propellant will be used up; the only case where one might save propellant is where it's needed for a return flight (the Moon missions) or, perhaps, in making visits to multiple planets. Then we can write the initial and final masses in terms of the three masses as

$$v_f = v_{ex} \ln\left(\frac{m_i + m_p + m_L}{m_i + m_L}\right)$$

$$= v_{ex} \ln\left(\frac{(m_p + m_L)(1+\alpha)}{(m_p + m_L)(1+\alpha) - m_p}\right)$$

We want to see how large the factor in parentheses can be made, so that we can reach, say, v_f=2 v_{ex}. So, ln(*stuff*) = 2, so that *stuff* = e^2 = 7.389.... We can go higher, but we soon reach a practical limit for several reasons.

Let's take the inverse of the *stuff* in parentheses, to get some simpler algebra. Then, we can do some algebra:

$$\frac{(m_p + m_L)(1+\alpha) - m_p}{(m_p + m_L)(1+\alpha)} = \frac{1}{e^2} \ (= 0.135...)$$

$$1 - \frac{m_p}{(m_p + m_L)(1+\alpha)} = \frac{1}{e^2}$$

$$\frac{mp}{(m_p + m_L)(1+\alpha)} = 1 - \frac{1}{e^2} = 0.865... \equiv Q$$

$$m_p = Q(m_p + m_L)(1+\alpha)$$

$$m_p(1 - Q(1+\alpha)) = Q(1+\alpha)m_L$$

We can now see how much mass of propellant we need for a given payload mass:

$$m_p = \frac{Q(1+\alpha)}{1 - Q(1+\alpha)} m_L$$

To appreciate what this means, let's take some numerical values. Take α=0.1; that is, the infrastructure can be cut to be only 10% of the total mass of propellant and payload. We've also specified Q = 0.865 so that the rocket can get to twice the exhausts velocity. Then,

$$m_p = \frac{0.865(1+0.1)}{1 - 0.865(1+0.1)} m_L$$

$$= 19.5 m_L \,!$$

The propellant outweighs the payload nearly 20-to-1! If we had, instead, α=0 (impossible), we could economize on propellant:

$$mp = \frac{0.865}{1 - 0.865} m_L$$

$$= 6.4 m_L$$

What a savings! Keep that infrastructure as light as possible, consistent with safety.

We can also see that there is an absolute limit on α, such that it's impossible to get to $v_f=2v_{ex}$ beyond that. You might solve for that value.

Peak kinetic energy

A rocket that burns up a very large fraction of its mass is, of course, smaller. At some point in the burn process, the rocket reaches its peak kinetic energy and then declines with further burning. For example, at 86% mass loss, we have $\ln(m_0/m_f) = 2$ and $v=2v_{ex}$. To get the $v=3v_{ex}$, the mass loss must be 95%. In the formula for kinetic energy, $K=mv^2/2$, the factor v^2 is now increased by a factor $3^2/2^2 = 2.25$, while the mass remaining has decreased by a factor (5/14) =1/2.8. Thus, the kinetic energy has decreased. At what point does kinetic energy reach its peak? We can use calculus to find out. We'll take the derivative with respect to mass (mass lost as propellant or mass remaining in the rocket, both equivalent) and find out where it has the value zero – the peak of the arc, where the rate of change of K with respect to mass has a horizontal slope and starts declining:

$$\frac{d}{dm_f}\left(\frac{m_f v^2}{2}\right)=\frac{d}{dm_f}\left(\frac{m_f}{2}\left[v_{ex}\ln\left(\frac{m_0}{m_f}\right)\right]^2\right)$$

$$=\frac{1}{2}\left[v_{ex}\ln\left(\frac{m_0}{m_f}\right)\right]^2+m_f v_{ex}^{\ 2}\ln\left(\frac{m_0}{m_f}\right)\frac{d}{dm_f}\ln\left(\frac{m_0}{m_f}\right)$$

The final factor can be resolved:

$$\frac{d}{dm_f}\ln\left(\frac{m_0}{m_f}\right)=\frac{1}{(m_0/m_f)}\frac{d}{dm_f}\left(\frac{m_0}{m_f}\right)$$

$$=\frac{m_f}{m_0}\left(-\frac{m_0}{m_f^{\ 2}}\right)$$

$$=-\frac{1}{m_f}$$

Here, I used (d/dy)ln(y)=1/y and the chain rule, (d/dx)ln(y(x))=(1/y)(dy/dx). Let's now finish:

$$\frac{dK}{dm_f}=\frac{v_{ex}^{\ 2}}{2}\left[\ln\left(\frac{m0}{m_f}\right)\right]^2+m_f v_{ex}^{\ 2}\ln\left(\frac{m0}{m_f}\right)\left(-\frac{1}{m_f}\right)$$

$$=\frac{v_{ex}^{\ 2}}{2}\ln\left(\frac{m0}{m_f}\right)\left[\ln\left(\frac{m0}{m_f}\right)-2\right]$$

This goes to zero if the factor in square brackets goes to zero, or, simply,

$$\ln\left(\frac{m_0}{m_f}\right)=2$$

which is to say that $m_f = e^{-2}\ m_0 = 0.135\ m_0$. This result is independent of the kind of propellant used or of the mass of infrastructure.

When is maximal kinetic energy the best criterion for rocket performance? Rarely... perhaps for rockets acting as kinetic "killers" of incoming missiles (or, more darkly, for destroying enemy navigation and reconnaisance satellites, as was demonstrated by China once, destroying one of its old satellites). Even then, the major goal of rocket flight is generally to get somewhere fast, not with the most kinetic energy. The result is then mostly amusing.

More germane to common use: when are rocket engines most efficient in adding kinetic energy to a rocket?

We'll evaluate this as the rate of gain of kinetic energy per mass of propellant consumed. We'll have similar derivatives to consider, but we'll keep the answer in terms of velocities (speeds, really, since we're not considering direction, yet).

$$\frac{dK}{dm_p} = \frac{d}{dm_p}\left[\frac{m_r}{2}v^2\right]$$
$$= \frac{v^2}{2}\frac{dm_r}{dm_p} + m_r v \frac{dv}{dm_p}$$

Here, m_r is the total mass of the rocket. Of course, since propellant consumption is the only way the rocket it losing mass, we have dm_r/dm_p = -1. We'll then use an earlier equation for v:

$$v_f = v_{ex} \ln\left(\frac{m_0}{m_f}\right)$$

which gives us

$$\frac{dv}{dm_p} = v_{ex} \frac{1}{(m_0 / m_f)} \frac{d}{dm_p}\left(\frac{m_0}{m_f}\right)$$

Now, that last derivative on the right side is the negative of the derivative with respect to m_r, so we get

$$\frac{dK}{dm_p} = -\frac{v^2}{2} + v v_{ex}$$
$$= \frac{v}{2}(2v_{ex} - v)$$

This has many interesting implications:

* At start-up, with v=0 (in the rocket's initial rest frame), v=0 and the efficiency of using propellant is a flat zero. (It's actually an efficacy, not an efficiency; the latter must be dimensionless, a pure number). This is much like discussions around the Froude number in aerodynamics. When is a helicopter or a jet using the least energy to keep hovering? It is when the biggest air mass is moving at the lowest possible speed. This is clearly not the case for a rocket with its high-velocity exhaust churning through still air. A side note here: even at its high exit velocity, the exhaust stream should be close to atmospheric pressure. That seems surprising. However, if it is not at that pressure, there are interesting instabilities and less thrust.

* When v = $2v_{ex}$, the efficiency is again zero. We know this already, from the exercise earlier which concluded that kinetic energy gain has stopped in this condition.

What makes a fast rocket = What makes a good propellant?

You would expect that it's one that releases the most energy per mass, and you'd be close. It's the propellant that generates the most *thrust* per unit mass, and that's not quite the same. Thrust per unit mass consumed is called *specific impulse, I_s* – really, thrust multiplied by the time, or impulse. You can see impulse declared on model rocket engines, such as from Estes Industries (which also does trucking, oddly – meaning that it's an odd combination of enterprises, not that they drive trucks oddly). Let's look at this. Consider thrust as velocity multiplied by mass per unit time, or

$$T = v_{ex}\frac{dm_p}{dt}$$

That makes thrust multiplied by the duration

$$Tt = v_{ex}\frac{dm_p}{dt}t$$
$$= v_{ex} m_p$$

and, very simply,

$$I_s = \frac{v_{ex} m_p}{m_p}$$
$$= v_{ex}$$

Let's consider an Estes rocket engine, of type B6. The B class designates a rocket with an impulse of 5 Newton-seconds (Ns) (and a B6 has mean thrust of 6 N, thus, a burn time of 5/6 = 0.83s; the engine has a following number, such as 4, in B6-4, meaning that it delays 4s before detonating a small charge to blow out a rocket-recovery parachute). The mass of propellant in such an engine is 6 g or 0.006 kg. Using metric units, we get

$$v_{ex} = \frac{I}{m_p} = \frac{5\,kg\,m\,s^{-2}\,s}{0.006\,kg}$$
$$= 833\,m\,s^{-1}$$

That's pretty speedy, about Mach 2.5, but other fuels are more remarkable. The Space Shuttle used liquid oxygen and liquid hydrogen, having a specific impulse of 4400 m s^{-1}. The highest specific impulse, 5320 m s^{-1}, was attained with a mixture of lithium, fluorine, and hydrogen, though it is rightly deemed not a practical fuel (Who wants hydrogen fluoride, which is the anhydrous form of glass-etching hydrofluoric acid, billowing out over the landscape, or any compound of very toxic lithium?). Here are a couple of interesting articles:

https://en.wikipedia.org/wiki/Liquid_rocket_propellant
https://en.wikipedia.org/wiki/Specific_impulse

The secret, though it really is not a secret but basic physical chemistry, of a high v_{ex} is, yes, a high energy of combustion, but also (1) low molecular mass, so that the denominator, mass, in specific impulse is small and (2) simple molecular composition. Let me amplify that last point a bit. We want as much energy as possible to go into the energy of translational motion of the exhaust products. Energy can also go into other modes of motion (degrees of freedom), rotation and vibration. The more complex the molecules, the more modes of rotation and vibration. Diatomic gases are best, having only 2 rotations and one vibration – so, HF is great, if impractical, being diatomic, being of very low molecular mass, and having a very high enthalpy of formation from an "aggressive" highly electronegative (electron-grabbing = oxidizing) fluorine and a rather electropositive hydrogen.

The exhaust gas is cooled by expansion, which helps get energy out of vibrational modes. There are good, if rather dense, explanations about the thermodynamics of rocket combustion, such as http://braeunig.us/space/thermo.htm.

Why have multistage rockets?

There are a couple of basic types of staging. One is using boosters that can be jettisoned, contributing thrust but then having their weight (mass) penalty rejected. Another is having one stage burn out and another one starting after the earlier stage is jettisoned. These can be paired, with boosters on a first stage that boosts a second stage. There are three-stage rockets. I haven't developed a presentation on the quantitative advantages of staging rockets. I may get to it sooner or later.

What about thrust vectoring?

Here we've been considering rockets whose thrust is all delivered along the axis of the rocket body. However, rockets always need to steer. Rockets are generally launched standing straight up, for several reasons, including getting out of the drag-producing atmosphere as fast as possible and avoiding the tendency to fall over (a big problem for very tall rockets). Yet, consider having to attain a trajectory that ends up parallel to the Earth's surface, as in docking with the International Space Station. The vertical flight has to be transformed into horizontal flight at the end. There are efficient and inefficient ways to do this. There is an optimal curvature of the trajectory that uses the least fuel, though not the least time, in such constructs as the Hohmann trajectory for transferring between two circular orbits at different altitudes. In any event, turning the rocket engine to move thrust off-axis reduces the efficiency of producing forward motion. I haven't worked up a presentation on this, but the nub of the idea is here.