(3) 9 Mar. 17

The exponential function in complex variables, and a geometric interpretation
— Vince Gutschick, again

We need a couple of concepts

First, complex variables.

They began with the puzzle of considering imaginary numbers, particularly $\sqrt{-1}$, often given the symbol i (for imaginary), or j.

After much "philosophizing," mathematicians took i to heart in creating a new part of math, complex variables, with numbers of the form $z = a + bi$, with both a and b as real numbers, a = real part = $\mathrm{Re}(z)$, b = imaginary part = $\mathrm{Im}(z)$.

Some interesting properties are:

Addition: $z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication: $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 + i b_1 a_2 + i a_2 b_1 + \underset{-1}{(i)^2} b_1 b_2$

$$= a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1)$$

Complex conjugate: change the sign of i:

$$\bar{z} = a - ib$$

$$z\bar{z} = (a + ib)(a - ib) = a^2 + b^2$$

or $|z| = \sqrt{z\bar{z}}$, where $|z|$ is the "magnitude" of z, its length as a vector in (x, y)

The multiplication property is quite useful. We have 2-dimensional numbers with "convenient" properties

We can define functions of complex variables, $w = f(z)$

Expanding the usual idea of differentiation, we'll call the derivative $f'(z)$ for a function termed analytic (having a derivative), if the derivative exists — that is,

$$f'(z) = \lim_{\varepsilon \to 0} \frac{f(z+\varepsilon) - f(z)}{\varepsilon}$$, no matter from which "direction" ε comes — as real, as imaginary, or any complex number

The existence demands some special properties.

Let's write $z = x + iy$, $f(z) = u(x, y) + i v(x, y)$. That is, the Re and Im part of $f(z)$ are both functions of the Re and Im parts of z

② 9 mar 17

Let ε be purely real:

$$f'(z) = \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon, y) - u(x,y) + i\left[v(x+\varepsilon, y) - v(x,y)\right]}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{u(x,y)} + \frac{\partial u}{\partial x}\varepsilon - \cancel{u(x,y)} + i\left[\cancel{v(x,y)} + \frac{\partial v}{\partial x}\varepsilon - \cancel{v(x,y)}\right]}{\varepsilon}$$

where $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ are "partial derivatives" of the real-valued function u and v — just $\lim_{\varepsilon \to 0} \frac{u(x+\varepsilon) - u(x)}{\varepsilon}$, for one

Often these derivatives are given shorthand notations,

$$\frac{\partial u}{\partial x} = u_x \qquad \frac{\partial v}{\partial x} = v_x$$

$$f'(z) = u_x + i v_x$$

Let ε be purely imaginary $= i\varepsilon$ — we have to get the same answer:

$$f'(z) = \lim_{\varepsilon \to 0} \frac{u(x, y+\varepsilon) - u(x,y) + i\left[v(x, y+\varepsilon) - v(x,y)\right]}{i\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{u(x,y)} + \frac{\partial u}{\partial y}\varepsilon - \cancel{u(x,y)} + i\left[\cancel{v(x,y)} + \frac{\partial v}{\partial y}\varepsilon - \cancel{v(x,y)}\right]}{i\varepsilon}$$

$$= -i u_y + v_y \qquad \left(\text{using } \frac{1}{i\varepsilon} = \frac{-i}{\varepsilon}\right)$$

The 2 results have to be the same:

$$f'(z) = u_x + i v_x = v_y - i u_y$$

$$\Rightarrow u_x = v_y \left(\text{or } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\right) \text{ and } v_x = -u_y \left(\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right)$$

There are lots of function like this, including the exponential.

Generalizing, we'll say e^z uses the same power series as e^x:

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \ldots$$

$$= 1 + (x+iy) + \frac{(x+iy)^2}{2} + \frac{(x+iy)^3}{6} + \ldots$$

$$= 1 + x + iy + \frac{x^2}{2} - \frac{y^2}{2} + \frac{2ixy}{2} + \frac{(x^3 + 3ix^2y - 3xy^2 - iy^3)}{6} + \ldots$$

$$e^z = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \frac{(x^3 - 3xy^2)}{6} + \dots \leftarrow u(x,y)$$

$$+ iy + ixy + i\frac{x^2 y}{2} - \frac{iy^3}{6} + \dots \leftarrow iv(x,y)$$

$$\frac{\partial u}{\partial x} = 1 + x + \frac{x^2}{2} - \frac{y^2}{2} + \dots$$

$$\frac{\partial v}{\partial x} = y\left[1 + x + \frac{x^2}{2} - \frac{y^3}{6} + \dots\right]$$

$$\frac{\partial u}{\partial y} = -y - xy + \dots = -y\left[1 + x + \dots\right]$$

$$\frac{\partial v}{\partial y} = 1 + x + \frac{x^2}{2} + \frac{y^2}{2} + \dots$$

We see $\frac{\partial u}{\partial x} = u_x = \frac{\partial v}{\partial y} = v_y$ ✓

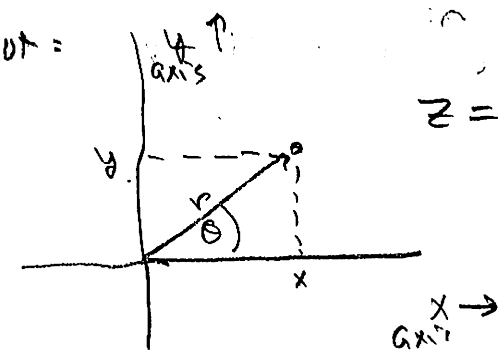
$\frac{\partial v}{\partial x} = v_x = -\frac{\partial u}{\partial y} = -u_y$ ✓

So e^z defined using the same power series as e^x in real variables is an analytic function in the complex plane (x, y)

There are many mathematically powerful properties of analytic functions that allow us to calculate integrals, solve differential equations, and more.

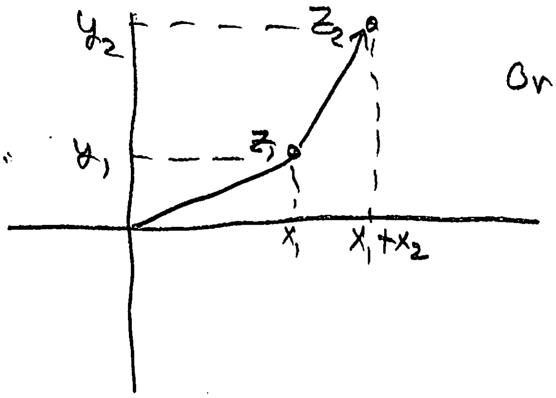
Let's just jump to a geometric interpretation of complex variables.

In a 2-dimensional plot:



$$z = x + iy = re^{i\theta}$$

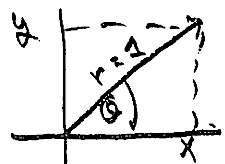
Aside, for now: complex numbers have the properties of vectors, in the way they add!



Or $\quad z_1+z_2 = x_1+x_2+i(y_1+y_2)$

Back to the task at hand. Consider a complex number of magnitude (modulus) 1, $\quad z = 1e^{i\theta} = e^{i\theta}$

The graphical view is



$\rightarrow \quad x = 1\cos\theta = \cos\theta$
$\qquad y = 1\sin\theta = \sin\theta$

This interpretation really works. Let's use the power series representation

$$z = e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{6} + \frac{(i\theta)^4}{24} + \frac{(i\theta)^5}{120} + \dots$$

$$= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots$$
$$+ i\theta - i\frac{\theta^3}{6} + i\frac{\theta^5}{120} + \dots$$

The real part, $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots$, is just the series expansion for $\underline{\cos\theta}$, as we'll see shortly.

The imaginary part, $\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} + \dots$, is $\underline{i\sin\theta}$

Or, $e^{i\theta} = \cos\theta + i\sin\theta$! --- just as the graph "says"

Showing that the series $1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} + \dots = \cos\theta$:

Any "nice" function such as $\cos\theta$ can be represented by a power series, here, a Maclaurin series in all its derivatives

$$f(x) = f(x)\Big|_{x=0} + f'(x)\Big|_{x=0} x + \frac{1}{2}f''(x)\Big|_{x=0} x^2 + \frac{1}{6}f'''(x)\Big|_{x=0} x^3 + \dots$$

(Valid if $f(x)$ is analytic around $x=0$)

f'' = 2nd derivative, f''' = 3rd derivative, etc.

I've claimed that, for $f(\theta) = \cos\theta$,

$f(0) = 1 \longrightarrow$ yes, $\cos(0) = 1$ ✓

$f'(0) = 0 \rightarrow f'(\theta) = \frac{\partial \cos\theta}{\partial\theta} = -\sin\theta$; $\sin 0 = 0$ ✓

$f''(0) = -1 \rightarrow f''(\theta) = \frac{\partial f'(\theta)}{\partial\theta} = -\cos\theta$; $-\cos 0 = -1$ ✓

etc.

We can do the same for $\sin\theta$

$f(0) = 0$

$f'(0) = \cos\theta\big|_{\theta=0} = +1$

$f''(0) = -\sin\theta\big|_{\theta=0} = 0$

$f'''(0) = -\cos\theta\big|_{\theta=0} = -1$

etc.

Another way to establish $e^{i\theta} = \cos\theta + i\sin\theta$, without using infinite series: from John W. Dettman, Applied Complex Variables, Macmillan, NY, 1965; the book used in the math course I loved most, Complex Variables, taught at Notre Dame 1965-66 by an engineer, whose name I really should remember! — see pages 52-53 there

The argument or development is based on a different definition of the exponential, that $\frac{df}{dz} = f(z)$ — its derivative equals its value, everywhere

(1) $f(z)$ is analytic (differentiable)

(2) $f'(z) = f(z)$

(3) Along the real line, $f(x) = e^x$

write $f(z) = u(x,y) + iv(x,y)$ for $z = x + iy$

From (1): $u_x = v_y$, $v_x = -u_y \rightarrow u_x + iv_x = v_y - iu_y$

From (2): $u_x + iv_x = u + iv$

Since $u_x = u$ and $v_x = v$, we have $u = e^x g(y)$, $v = e^x h(y)$ — that is, the functions u and v are separable in variables x and y.

Now, $v_y = u_x = u \Rightarrow h'(y) = g(y)$ and $-u_y = v \Rightarrow -g'(y) = h(y)$.

Differentiate once more

$-g''(y) = h'(y) = g(y)$ — hmmm: $\sin y$ and $\cos y$ look like this

Keep going::

$g(y) = A\cos y + B\sin y$, in general. We'll solve for the numerical values of A and B, once we get a second equation.

$h(y) = -g'(y) = --A\sin y - B\cos y = A\sin y - B\cos y$

Since $u(x,0) = e^x g(0) = e^x(A\cos 0 + B\sin 0) \doteq e^x$

$\rightarrow A e^x = e^x$

We have $A=1$

Since $v(x,0) = e^x h(0) \Rightarrow h(0)=0$, we have $A\sin 0 - B\cos 0 = 0$

No imaginary part; $\sin 0 = 0$; requires $B=0$

Thus, $f(z) = e^z = e^x\cos y + ie^x\sin y$

This is, with a change of notation, as $z = xe^{i\theta}$, and $e^z = e^x e^{i\theta}$

$e^z = e^x\cos\theta + ie^x\sin\theta = e^x(\cos\theta + i\sin\theta)$

$\rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

– the same result as the other method yielded

One useful relation with a geometric interpretation is

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = (r_1 r_2) e^{i(\theta_1+\theta_2)}$$

The relation $e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ comes from the standard rule for products of numbers with exponents, $a^{n_1} a^{n_2} = a^{n_1+n_2}$, with $a = e$ and $n_1 = i\theta_1$, $n_2 = i\theta_2$

Multiplying 2 complex numbers is multiplying their magnitudes and adding their angles. This is useful in many calculations, such as in electronic circuit analysis, which I won't go into here.

There are other cool things about complex variables, including that every analytic function $f(z) = u(x,y) + iv(x,y)$ is a solution of a famous differential equation used in physics, $u_{xx} + v_{xx} = 0$, the Laplace's equation, involved in electromagnetic theory, heat transfer, and with extension to Poisson's equation (in 2 variables, written $\nabla^2 f = g$), to gravity and more. The proof, a short one, is on page 42 of Dettman's book.