Skydiving – as an exercise in calculus   Vince Gutschick, Ph. D.   18-19 Sept. 2017

A fair-sized body (e.g., a human skydiver) falling in air at normal air density near the surface accelerates rapidly under the force of gravity, at first:

Rate of gain of downward momentum, $m\frac{dv}{dt} = -mg$, or $\frac{dv}{dt} = -g$ ρ

The solution is trivial; taking initial velocity as zero (diving from level flight), v=-gt.
In the first second, one attains a downward velocity, -9.8 m/s; in the 2nd second, -19.6 m/s, ...

The distance fallen is easily computed from $h = \int_0^t dt'(-gt') = -g\frac{t^2}{2}$. In the first second, one falls 4.9 m; in the next second, one has fallen 19.6m, in the third second, 44.1 m, and so on.

However, there is a drag force from moving through the air, in the opposite direction from one's velocity, pushing upward and reducing the acceleration. The drag arises from pressure drag, moving air out of the way, from the acceleration of that air, and from skin drag of air flowing over the side of the body and dissipating energy from viscous damping. The net force is closely proportional to v^2 for bodies the size of a human body in air:

$$F = F_{grav} + F_{drag} = -mg + \frac{1}{2}A\rho C_d v^2$$

Here, *A* is the effective frontal area of the skydiver's body, which depends upon the way the person is oriented, whether limbs are extended, and how the body is "balled up" or not. We'll take it in the calculations below as 0.2 m^2; ρ is the density of air in SI units (kg m^{-3}); C_d is the coefficient of drag, a dimensionless number, near 1.0 in our case, though declining slowly with increasing speed.

At a high enough velocity, the terminal velocity, v_t, drag balances gravity and acceleration ceases; the velocity is steady:

$$\frac{1}{2}A\rho C_d v_t^2 = mg, \quad \text{or} \quad v_t^2 = \frac{2mg}{A\rho C_d}$$

Let's pick m = 70 kg (the silly old notion of a standard male; skydivers are also female, of course). This give us

$$v_t^2 = \frac{2*70kg*9.8m\,s^{-2}}{0.2m^2*1.3kg\,m^{-3}*1.0} = 5277\,m^2s^{-2}$$

$$\rightarrow vt = 73\,m\,s^{-1} \;-\; 160\,mph!$$

Let's assume a careful skydiver, who will pull a parachute open. How fast does he or she approach terminal velocity, and how far does he or she fall by the time of reaching a given fraction of terminal velocity?

We have to solve the differential equation

$$\frac{dv}{dt} = -g + \frac{A\rho C_d}{2m}v^2 = -g + g\frac{v^2}{v_t^2} = -g\left(1 - \frac{v^2}{v_t^2}\right)$$

We can solve this by separation of variables, bringing terms in v to the left-hand side:

$$\frac{dv}{1-v^2/v_t^2} = -g\,dt$$

Now change variables. Let x = v/v_t, dx = dv/v_t, dv = $v_t dx$:

$$v_t \frac{dx}{1-x^2} = -g\,dt$$

Integrate both sides, while moving v_t to the right-hand side:

$$\int\left(\frac{dx}{1-x^2}\right) = -\frac{g}{v_t}t$$

We'll worry about the limits of integration in a minute. What we have to see now is that the integral on the left is simply atanh(x), the inverse hyperbolic tangent of x, or the hyperbolic arctangent. We'll derive that, and also show that it's straightforward to use the hyperbolic functions from their definitions.

Just as for the circular or geometric functions, we have sines, cosines, and tangents in their equivalents.

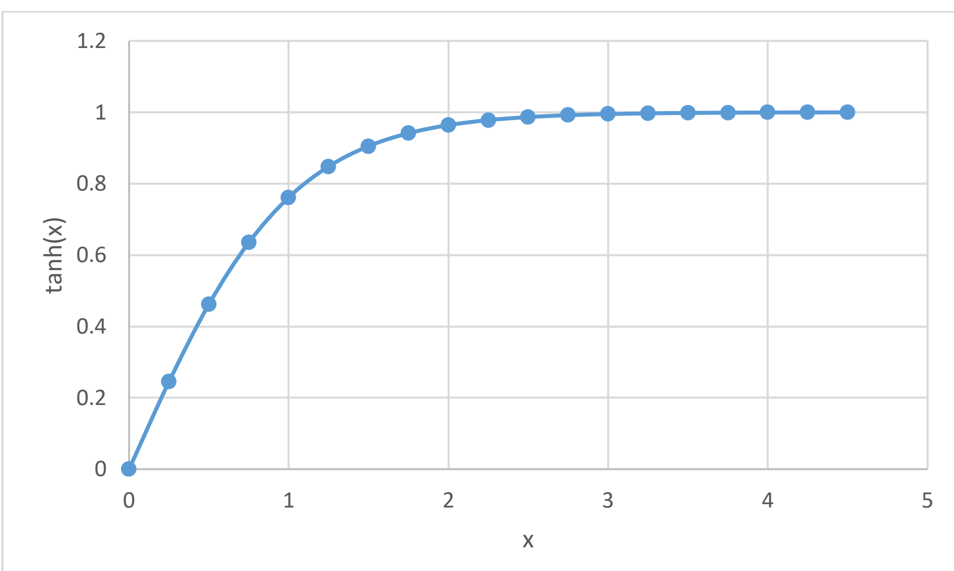
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$$

Their definitions are a bit interesting:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \quad \cosh(x) = \frac{e^x + e^{-x}}{2}$$

We can see that, at small x, sinh(x) looks like x and smaller terms; cosh(x) starts at 1, but has smaller terms that are increasing, vs. the case for sin(x).

Here's what tanh(x) looks like:



The asymptotic value of 1 at large x is easily seen; the e^{-x} term becomes tiny compared to e^x and we have tanh(x) → e^x/e^x = 1. So, too, will our velocity flatline.

To show that the new integral is really atanh(x), let's start by differentiating both sides of this equation:

$$\int^{x} \frac{dx'}{1-x'^2} = \text{atanh}(x)$$

Take d/dx of both sides:

$$\frac{1}{1-x^2} = \frac{d}{dx}\text{atanh(x)}$$

To evaluate the derivative on the right, let's write

$$y = \text{atanh(x)}$$

$$\rightarrow x = \tanh(y)$$

$$\rightarrow dx = \frac{d}{dy}\tanh(y)dy$$

It gets a bit lengthy from here. First, we're going to show that (d/dy)tanh(y) = 1/cosh^2(y):

$$\frac{d}{dy}\text{tanh(y)}=\frac{d}{dy}\frac{\sinh(y)}{\cosh(y)}$$

By the rule for the derivative of a ratio, this is

$$\frac{\frac{d}{dy}\sinh(y)*\cosh(y)-\sinh(y)*\frac{d}{dy}\cosh(y)}{\cosh^2(y)}$$

Now,

$$\frac{d}{dy}\sinh(y) = \frac{d}{dy}\frac{e^y-e^{-y}}{2} = \frac{e^y-(-e^{-y})}{2} = \cosh(y)$$

It's also trivial to show, simlarly, that

$$\frac{d}{dy}\cosh(y) = \sinh(y)$$

So, we have

$$\frac{d}{dy}\tanh(y) = \frac{\cosh^2(y)-\sinh^2(y)}{\cosh 2(y)}$$

It's also simple to show that the cosh^2(y)+sinh^2(y0 = 1:

$$\frac{e^{2y}+2+e^{-2y}}{4} - \frac{e^{2y}-2+e^{-2y}}{4} = \frac{4}{4} = 1$$

We have our proof.

To continue,

$$dx = \frac{1}{\cosh^2(y)}dy \;\rightarrow\; \frac{dy}{dx} = \cosh^2(y) = 1+\sinh^2(y)$$

Divide both sides by cosh^2(y):

$$
\begin{aligned}
\frac{1}{\cosh^2(y)}\frac{dy}{dx} \equiv 1 &= \frac{1}{\cosh^2(y)} + \frac{\sinh^2(y)}{\cosh^2(y)} \\
&= \frac{1}{dy/dx} + \tanh^2(y) \\
&= \frac{1}{dy/dx} + x^2
\end{aligned}
$$

$$
\rightarrow \frac{1}{dy/dx} = 1 - x^2
$$

$$
\rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}
$$

That's what we wanted to show! Q.E.D.

Now we can say

$$
-\frac{gt}{v_t} = \text{atanh(x)=atanh}\left(\frac{v}{v_t}\right)
$$

Let's take tanh of both sides:

$$
\tanh\left(\frac{gt}{v_t}\right) = -\frac{v}{v_t}
$$

Or

$$
v = -v_t \tanh\left(\frac{gt}{v_t}\right)
$$

At last, we have an explicit, analytic form for velocity. Note that v is negative, since the skydiver is falling down. It also has the right limits. At small values of t, $\tanh(gt/v_t) \rightarrow gt/v_t$, so that $v \rightarrow -gt$. At large values of t, tanh → 1 and $v \rightarrow v_t$.

Let's look at some concrete examples – how long does it take to get to ½ terminal velocity, or 90%, or more? How far has one fallen by then?

For any fraction, f, of terminal velocity, we require $f = \tanh(gt/v_t)$. Look up tanh values in a table and finr the argument, $a = gt/v_t$. Then, $t = a\, v_t/g$. To get the height, h, that one has fallen in that time, we have to integrate the velocity over time,

$$
h = \int_0^t dt'v(t') = -v_t\int_0^t dt'\tanh(gt'/v_t)
$$

Now, it's very easy to show that

$$
\int ds \tanh(s) = \ln(\cosh(s))
$$

since

$$\int ds \tanh(s) = \int \frac{ds \sinh(s)}{\cosh(s)} = \int \frac{d \cosh(s)}{\cosh(s)}$$
$$= \ln(\cosh(s))$$

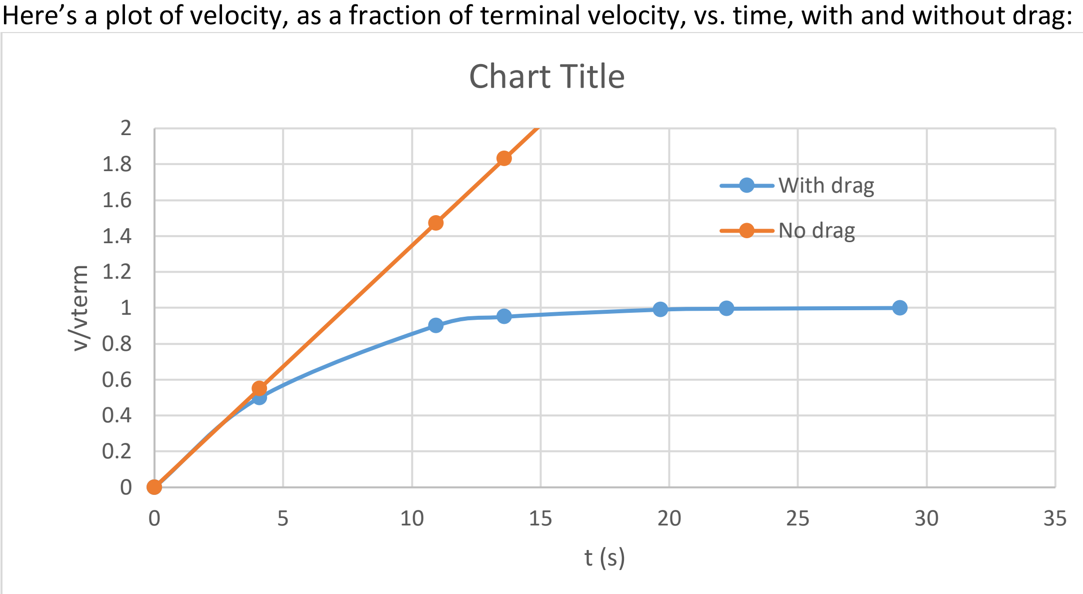
Then, with the change of variables s=gt/v_t, dt = (v_t/g)ds, we have

$$h = -\frac{v_t^{\,2}}{g} \ln(\cosh(gt / v_t))$$

So, Choose the fraction, f; find the argument, a, such that tanh(a)=f; thus, find t; for the height, h<0, with the formula above. We can make a table

f	a	t	cosh(a)	ln(cosh(a))	h	h, no drag	v/vt, no drag
0	0	0	1	0	0	0	0
0.50000	0.55	4.08	1.15	0.14	-78	-82	0.55
0.90000	1.47	10.93	2.29	0.83	-449	-586	1.47
0.95000	1.83	13.60	3.20	1.16	-629	-907	1.83
0.99000	2.65	19.66	7.09	1.96	-1059	-1893	2.65
0.99500	2.99	22.24	10.01	2.30	-1245	-2423	2.99
0.99918	3.90	28.96	24.71	3.21	-1734	-4111	3.90

Here's a plot of velocity, as a fraction of terminal velocity, vs. time, with and without drag:



We've done a couple of simplificiations. One, already noted, is that there is a slight dependence of the coefficient of drag on speed. A second, more serious for long falls, is that the atmosphere is thinner where the dive has to start. A fair estimate of density vs altitude, z, is

$$\rho(z) = \rho_0 e^{-z/8800m}$$

Presuming that the drop of, say, 1893 m has to start at about 2400 m to give time for the parachute to slow the diver down, then the initial density is only 0.76 as large as at sea level. The drag is less, in proportion, and the fall is faster. There is no analytical solution for t, f, etc. with drag as a function of h (or, z-h, really). There can be some clever approximations, but I won't pursue them here.

If you look online for "terminal velocity," you will often find a cruder approximation for drag, going linearly with speed vs. quadratically. Suppose we set the same terminal velocity, -73 m/s, but use the simpler equation,

$$\frac{dv}{dt} = -g\left(1+\frac{v}{v_t}\right)$$

Why the "+" sign in front of v/v_t? It's because v is negative; this is then a moderating effect on acceleration, just as was $-(v^2/v_t^2)$ in the more realistic case.

This equation is easy to solve, again by separation of variables:

$$\frac{dv}{1+\frac{v}{v_t}} = -g\,dt$$

A simple change of variables to $x=v/v_t$ gives us

$$\frac{dx}{1+x} = -\frac{g}{v_t}dt$$

and then, with the change y = 1+x, we get

$$\frac{dy}{y} = -\frac{g}{v_t}dt$$

and

$$\ln(y)\Big| = -\frac{gt}{v_t}$$

Taking the exponential of both sides, we get

$$1+\frac{v}{v_t}-1 = \frac{v}{v_t} = e^{-gt/v_t}-1$$
$$\rightarrow v = -v_t[1-e^{-gt/v_t}]$$

This is nice simple "relaxation" equation. How does it compare with the more accurate one? We can make another table. With $1-\exp(-gt/v_t)$ as f, the same fraction of terminal velocity we used before, we can solve for t.

$$f = 1-e^{-gt/v_t}$$
$$e^{-gt/v_t} = 1-f$$
$$-\frac{gt}{v_t} = \ln(1-f)$$
$$t = -\frac{v_t}{g}\ln(1-f)$$

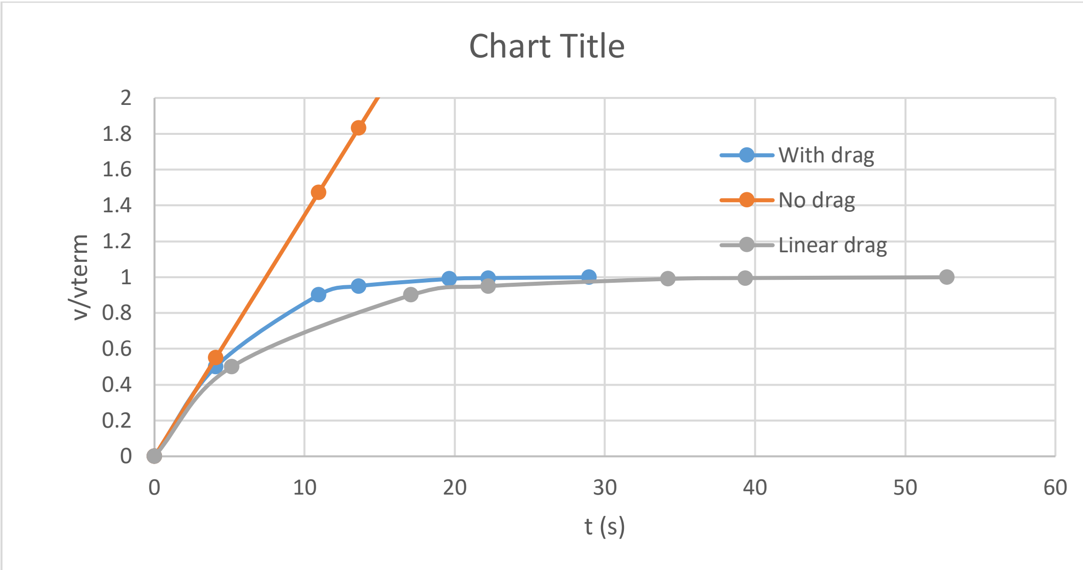
We can integrate v(t) over time to get the fall, using

$$\int_0^s ds'e^{-s'} = e^{-s}-1$$

Then,

$$\int_0^d dt'v(t') = -v_t t + \frac{v_t^2}{g}\left(1-e^{-gt/v_t}\right)$$

Let's add this estimation of velocity vs. time to the first chart:



Oddly, the skydiver has to slow down faster, in order to match the actual terminal velocity computed using the quadratic drag.

We can also plot height fallen vs. time for the two cases – well, three: realistic quadratic drag, linear drag, and no drag: