A study in predatory lending, using calculus

At our home we received a flier today, 6 April 2017, from Buddy's home Furnishings. It was largely in Spanish, clearly targeting Hispanics and touting that the buyer need not have a credit rating. Red flags!

Each item, such as a big TV, had its weekly or monthly payment rate and the duration of payments. By law, I guess, the flier also had to specify the cost for buying it outright...and that was typically *half* the cost on the installment plan that might be 2 years or even as short as 1.25 years (65 weeks)! That's a whopping interest rate, on the order of 100%.

We can get a good estimate of the interest rate, *r*, changing from a discrete set of payments (with a somewhat clumsy series sum) to a continuous payment rate, creating a differential equation that's rather readily solved.

In the general case, we have

P_0 = the principal of the loan (the one-time purchase price)

C' = the payment rate (dollars per week or month, or, better, per year to get the annual rate)

t = the duration of payments on a stated plan, commonly 1.25 or 2 years

We have to find the interest rate, *r*

If the payment rate is high enough, the principal declines, eventually to zero. In any one payment in a time Δt, the principal increases by an amount Pr Δt but decreases by the amount C' Δt. Then we have

ΔP = rP Δt-C' Δt

Taking the limit of small Δt, we get a differential equation

$$\frac{dP}{dt} = rP - C'$$

We can solve it in steps. First, the homogeneous equation, dP/dt = rP, has the simple solution

$$P = P_0 e^{rt}$$

The inhomogeneous equation has the solution

$$P = P_0 e^{rt} - C' \int_0^t dt' e^{r(t-t')}$$

You can verify this by differentiating it with respect to t.

We can integrate this to a closed form

$$\begin{aligned} P &= P_0 e^{rt} - C' e^{rt} \int_0^t dt' e^{-rt'} \\ &= P_0 e^{rt} - C' e^{rt} \frac{e^{-rt'}}{-r} \Big|_0^t \\ &= P_0 e^{rt} + \frac{C'}{r} e^{rt} \left[e^{-rt} - 1 \right] \\ &= e^{rt} \left[P_0 - \frac{C'}{r} \left(1 - e^{-rt} \right) \right] \end{aligned}$$

We have to solve for the term in brackets reaching zero. Let's divide out P_0:

$$1 = \frac{C'/P0}{r}\left[1-e^{-rt}\right]$$

Now consider the example of a big TV, with a full price P_0 = \$1560, a term of 2 years, and a weekly payment of \$30 or a payment rate per year of \$1560...of course, the total payments are then twice the one-time purchase price! We have $C'=P_0$ or $C'/P_0 = 1$. The specific equation to solve is then

$$1 = \frac{1}{r}\left[1-e^{-2r}\right]$$

It's simple enough but nonlinear. Let's try two ways to solve it numerically:

(1) Iterated guesses. The interest rate is not quite 100% per year. I chose an initial guess of 0.60 = 60%:

With r=0.60:

$$1 = ?\frac{1}{0.6}\left[1-e^{-1.2}\right] = 1.168 \quad \textit{Not yet close enough}$$

(I couldn't get the question mark above the equals sign)

Try r =0.65

$$1 = ?\frac{1}{0.65}\left[1-e^{-1.3}\right] = 1.119 \quad \textit{Take a bigger step now}$$

Try r=0.75

$$1 = ?\frac{1}{0.75}\left[1-e^{-1.5}\right] = 1.036 \quad \textit{Getting close}$$

Try r=0.80

$$1 = ?\frac{1}{1.6}\left[1-e^{-1.6}\right] = 0.998 \quad \textit{Close enough}$$

That's 80% per year, compounded!

Why would anyone take that rate, when even usurious credit card rates are about 27% (and they are usurious because South Dakota refuses to pass a usury law, so that they can get the credit card business)? Easy: the prospective buyers have no credit rating, so this is the only way they can buy a gib-ticket item. What's more, they will be caught in a cycle, which can get worse if they miss a payment and get a big penalty. The only thing in defense of the lenders is that there is probably a significant rate of loss, being unable to repossess the item in good shape.

(2) Using calculus twice

We're trying to find the root of the equation

$$F(r) = 1 - \frac{1}{r}\left[1-e-2r\right]$$

We make a guess, calling it $r=r_0$. We evaluate $F(r_0)$. Then, we assume that the function F(r) is nearly linear near $r=r_0$, so that at another nearby value r_1, we have

$$F(r_1) \approx F(r_0) + F'(r_0)(r_1 - r_0)$$

Here, F' is the derivative of F with respect to r. Setting $F(r_1)=0$, as we hope, we get

$$r_1 = r_0 - \frac{F(r_0)}{F'(r_0)}$$

Nice and simple. This is Newton's rule (*the* Newton).

In our current specific case, we can readily calculate the derivative:

$$\begin{aligned} F'(r) &= +\frac{1}{r^2}\left[1-e^{-2r}\right] - \frac{1}{r}2e^{-2r} \\ &= \frac{1}{r^2} - \left(\frac{1}{r^2}+\frac{2}{r}\right)e^{-2r} \end{aligned}$$

Let's start with the same guess as before, r=0.6:

$$\begin{aligned} F(r_0) &= 1-1.168 = -0.168 \\ F'(r0) &= \frac{1}{0.6^2} - \left(\frac{1}{0.6^2}+\frac{2}{0.6}\right)e^{-1.2} \\ &= 0.937 \\ \rightarrow r_1 &= 0.6 - \frac{-0.168}{0.937} \\ &= 0.779 \\ F(r1) &= 1-1.013 \quad (\mathit{Work\ not\ shown - easy\ to\ do}) \\ &= -0.013 \quad \mathit{Getting\ close\ fast} \end{aligned}$$

A second iteration gives us

$$\begin{aligned} F'(r_1) &= \frac{1}{0.779^2} - \left(\frac{1}{0.779^2}+\frac{2}{0.779}\right)e^{-1.558} \\ &= 0.760 \\ \rightarrow r_2 &= 0.779 - \frac{-0.013}{0.760} \\ &= 0.796 \\ F(r_2) &= 1-1.001 = -0.001 \quad \mathit{Close\ enough!} \end{aligned}$$

Being on a roll, let's try a 2^{nd} case, a furniture set. The one-time purchase price is P_0 = \$553. The weekly payment, in one plan, is \$17, and the term is 65 weeks or 1.25 years, for a total price of \$1105 = $2P_0$...and this on a shorter loan period, so the interest rate is notably higher than 80%. Let's see how high.

We have $C'/P_0 = (2P_0/1.25)/P_0 = 1.6$, and t=1.25 in years. The equation is then

$$F(r)=\frac{1.6}{r}\left[1-e^{-1.25r}\right]$$

$$and$$

$$F'(r)=\frac{1.6}{r^2}-\left(\frac{1.6}{r^2}+\frac{2}{r}\right)e^{-1.25r}$$

Without showing the details of the calculations:
First guess:

r_0=1.000 (100% interest)

$F(r_0)$=-0.142, $F'(r_0)$ = 0.569

→ r_1 = 1.250

$F(r_1)$=-0.012, $F'(r_1)$=0.474

→ r_2 = 1.275

F(r_2 = 1.000 to 3 places (0.99995) – got it

So, this is 127.5% interest!