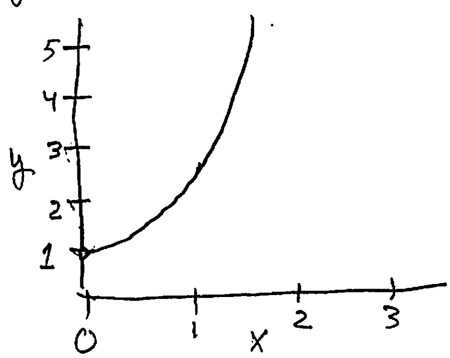
Two interesting functions that are also inverses of each other and that come up naturally in the sciences:

- the exponential, e^x – pronounced as "e to the x"
- the natural logarithm, $\ln x$ – pronounced as "el en x" or "log x"

e^x is a function describing growth at compound interest = growth at a rate proportional to it current (increasing) size

$y = e^x$ is a solution of $\frac{dy}{dx} = y$



Note that $e^0 = 1$ (any base to the zero power equals 1)

$e = 2.71828\ldots$, an irrational number

Analogy of compound interest:

y starts at y_0 and after an interval Δt, it increases by r (fractional interest rate) multiplied by Δt

$y_1 = y_0 (1 + r\Delta t)$ – call it $y_0 (1+a)$, $a = r\Delta t$

$y_2 = y_1 (1+a) = y_0 (1+a)^2$

$y_3 = y_2 (1+a) = y_0 (1+a)^3$

Multiply it out

$$y_n = (1+a)^n = 1 + na + \frac{n(n-1)}{2} a^2 + \frac{n(n-1)(n-2)}{6} a^3 + \ldots$$

Let's look at $a = r\Delta t$ as $r\frac{t}{n}$, where t is the total time,
$t = n\Delta t$, with Δt being the compounding time.
Let's make this a very small interval — getting continuous compounding:

$(1+a)^n$ becomes $\left(1+\frac{rt}{n}\right)^n$

$$y_n = 1 + n\frac{rt}{n} + \frac{n(n-1)}{2}\left(\frac{rt}{n}\right)^2 + \frac{n(n-1)(n-2)}{6}\left(\frac{rt}{n}\right)^3 + \ldots$$

As $n \to \infty$: $y \to 1 + (rt) + \frac{(rt)^2}{2} + \frac{(rt)^3}{6} + \ldots$

$$\equiv e^{rt}$$

This gives us a definition of e^x as an infinite series
(which always converges to a finite answer for any $x < \infty$):

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$$

and the numerical value of e:

$$e = e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots = 2.71828\ldots$$

We can take e as the base of a number system, just like
we ordinarily take 10 (in the decimal system)
or 2 (in the binary system).

Now, in any number base system, we define a logarithm function.
Thus, for 10^n = some number, n is the base-10 logarithm,
the power to which 10 must be raised to give that number.
So, in the base-e system, $\ln x$ is the natural logarithm of x:

$$x = e^{\ln x}$$

Do we have a way to compute $\ln x$ for any x? Yes:

$$\frac{d e^{\ln x}}{dx} = \frac{dx}{dx} = 1$$

But we can use the chain rule: $\frac{df(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$

$$\frac{d e^{\ln x}}{dx} = \frac{d e^{\ln x}}{d(\ln x)} \frac{d(\ln x)}{dx}$$

Another interesting property of e^x is:

$$\frac{de^x}{dx} = e^x \quad \text{itself}$$

You can show this by differentiating the series expression term-by-term

You can also see it from the relation that $e^x = y$ is the solution of the equation $dy = y\,dx$ or $\frac{dy}{dx} = y$

So, we have

$$1 = \frac{de^{\ln x}}{d(\ln x)} \frac{d(\ln x)}{dx} = e^{\ln x} \frac{d(\ln x)}{dx} = x \frac{d \ln x}{dx}$$

$$\rightarrow d \ln x = \frac{dx}{x} \rightarrow \ln x_f - \ln x_0 = \int_{x_0}^{x_f} \frac{dx}{x}$$

Start at $x_0 = 1$: $\ln 1 = 0$ (that is, $e^{\ln 1} = 1 = e^0 \Rightarrow \ln 1 = 0$

So, $\ln x = \int_1^x \frac{dx}{x}$

We can also see that $\ln x_f - \ln x_0 = \ln \frac{x_f}{x_0}$ (work this out)

Now we have two useful functions

You can try evaluating e^x and $\ln x$ numerically, say, in Excel

e^x: Sum the series ... and see how many terms you had to use until the added terms stop making a difference.

Try $e^{0.5}$; $e^{0.693}$, e^1, e^2

↓ will be interesting

$\ln x$: Set up the integral as a Riemann sum, using a step $\Delta x = 0.01$, say.

Try $\ln 2$, $\ln 3$, $\ln 6$

↳ this will be $\ln 3 \times 2 = \ln 3 + \ln 2$

Also easy in Excel

Start at $x_1 = 1 + \frac{1}{2}\Delta x = 1.005$ — compute $\frac{\Delta x}{x} = \frac{0.005}{1.005}$

Go to $x_2 = x_1 + \Delta x = 1.015$ — compute $\frac{\Delta x}{x} = \frac{0.005}{1.015}$

Or, simply, multiply by Δx at the end, & compute

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \dots \frac{1}{x_{100}}$$

↳ Gets you to $x = 2$

I got the answer 0.693144... in a short Python program:

```
# lnx.py
dx = 0.01
y = 0.
x = 1.005
for n in range (0,100):
    y = y + 1/x
    x = x + dx
y = y*dx
print y
```

The exact answer is 0.693147 !