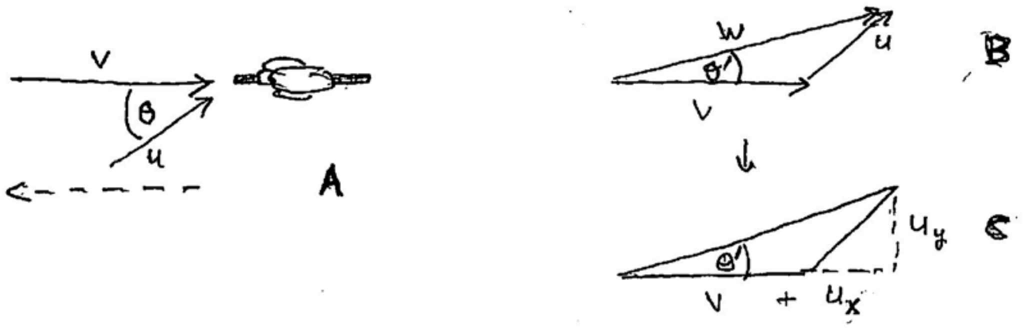
What a drag the wind is

Riding a bike into a headwind adds notably to our effort and reduces our speed. A tailwind is helpful, on the other hand. Is a crosswind, exactly perpendicular to our direction of travel, neutral, that is, not changing our effort from the case of no wind? We need to look at the physics of air drag to figure this out. We'll get even more details of interest as we consider various wind directions. Let's diagram our conditions:



Consider a cyclist, as in part A of the figure, heading at speed *v* to the left. His or her motion generates a headwind component of magnitude *v* moving in the positive-x direction (x is the horizontal axis). There's also a wind of magnitude *u* blowing at an angle ϑ off of the headwind direction. (Note that ϑ can be greater than π/2 radians (90°), representing a tailwind. It can also be negative, blowing from the other side, though the effect on drag is the same.)

Now see part B of the figure. The net wind on the cyclist is a vector, $\vec{w}$ made up of the two vectors specified by *v* and *u*. If you're used to vector notation (which is not necessary), we can say

$$\vec{v} = v\vec{e}_x,\ \vec{u} = u_x\vec{e}_x + u_y\vec{e}_y = u\cos\theta\vec{e}_x + u\sin\theta\vec{e}_y,$$

where $\vec{e}_x$ is the unit vector in the x-direction and $\vec{e}_y$ is the unit vector in the y-direction. In any case, the resultant wind has a component $w_x = v + u_x$ in the x-direction and a component $w_y = u_y$ in the y-direction. Geometry tells us that the net wind blows on the cyclist at an apparent angle θ'; from part C of the figure, we calculate simply

$$\tan\theta' = \frac{u_y}{v+u_x} = \frac{u\sin\theta}{v+u\cos\theta}$$

Now we're ready to calculate the drag force, which depends upon the *square* of the effective windspeed *w*. We can also calculated what fraction of the force acts in the x-direction, against the cyclist – the drag exerted perpendicular to the cyclist's path just has to be balanced by leaning; no work is done on or by the cyclist because there's no movement of the cyclist in the y-direction.

The drag force, *F*, at the speed of a cyclist, is well represented by the formula

$$F = \frac{1}{2}C_d\rho w^2$$

Here, C_d is the drag coefficient that depends upon the size and shape of the bike plus cyclist, and we'll take it as a constant, independent of windspeed. That's a good approximation; for more details, there are fascinating stories about all kinds of physical phenomena in Mark Denny's book, *Air and Water.* In the formula, ρ is the density of air, again a constant. Since we're only interested in relative forces as they depend upon the magnitudes of v and u and the heading angle ϑ, we'll just compare the magnitude of w^2 as it depends on these three quantities. Let's rename $F = w^2$ as a kind of (unit-)reduced force. We have the right-triangle relation that $w^2=w_x^2+w_y^2$, or

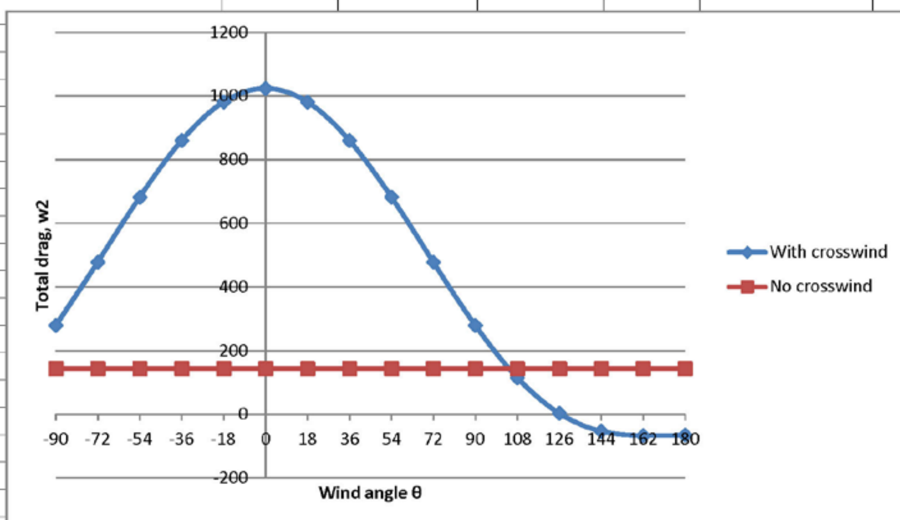
$$\begin{aligned} w^2 &= (v+u\cos\theta)^2 + u^2\sin^2\theta \\ &= v^2 + 2vu\cos\theta + u^2\cos 2\theta + u^2\sin 2\theta \\ &= v^2 + 2vu\cos\theta + u^2 \end{aligned}$$

I used the simple relation that $\sin^2\vartheta + \cos^2\vartheta = 1$.

The part of the "force," w^2, that acts against the progress of the cyclist is $w^2\cos\vartheta'$.

To explore the effects of different speeds and wind directions, we'll go to numerical calculations in a spreadsheet. The spreadsheet is linked here. I've also added a printout of one run with the spreadsheet, for a high windspeed u and a modest cyclist ground speed, v, over a range of wind angles, ϑ. You can explore other combinations in the spreadsheet by changing the entries in cells C1 and C2.

v		12	pi/2=	1.570796			
u		20					
		wx=	wy=	w			
theta	theta(deg.)	v+u cos(theta)	u sin(theta)		theta'=atan(wy/wx)	F=h^2 cos(theta')	v^2 only
-1.570796327	-90	12	-20	23.32381	-1.030376827	279.885691	144
-1.256637061	-72	18.18033989	-19.0211303	26.31213	-0.807995317	478.3634569	144
-0.942477796	-54	23.75570505	-16.1803399	28.7426	-0.597937927	682.8006726	144
-0.628318531	-36	28.18033989	-11.755705	30.53405	-0.395211234	860.4598831	144
-0.314159265	-18	31.02113033	-6.18033989	31.63079	-0.196655069	981.2229824	144
0	0	32	0	32	0	1024	144
0.314159265	18	31.02113033	6.180339887	31.63079	0.196655069	981.2229824	144
0.628318531	36	28.18033989	11.75570505	30.53405	0.395211234	860.4598831	144
0.942477796	54	23.75570505	16.18033989	28.7426	0.597937927	682.8006726	144
1.256637061	72	18.18033989	19.02113033	26.31213	0.807995317	478.3634569	144
1.570796327	90	12	20	23.32381	1.030376827	279.885691	144
1.884955592	108	5.819660113	19.02113033	19.8915	1.273882811	115.7617794	144
2.199114858	126	0.244294954	16.18033989	16.18218	1.555699215	3.953225898	144
2.513274123	144	-4.180339887	11.75570505	12.47685	1.912452117	-52.15748324	144
2.827433388	162	-7.021130326	6.180339887	9.353762	2.419797737	-65.67398528	144
3.141592654	180	-8	2.4503E-15	8	3.141592654	-64	144



The rightmost column indicates the magnitude of the drag without any external wind. Note that the drag on the cyclist is always greater than that – in fact, twice as much, even with the wind purely crosswise. How does this happen? You have to consider that drag generates eddies or circular flows of air. When the cyclist is moving in still air, they have a certain pattern. When the cyclist is moving in a crosswind, the eddies change their structure in the new net wind field.

We can go on, to figure out the cyclist's maximal speed at various windspeeds u and heading angles ϑ. The idea is that the cyclist has a maximal power output, working against wind drag plus mechanical drag (as in gear and wheel friction). Normally, the former is more important. Let's make some estimates:

* Assume the cyclist can hit a speed v_0 in still air. He or she is then overcoming two dissipative forces:
* Call the mechanical force F_m and the wind drag force F_v.
* A reasonable assumption is that the mechanical forces scale in proportion of v, so we'll write
  $F_m = Mv$, with M as an undetermined coefficient.
* We know that the drag force scales as w^2, so we'll write it as

$F_v = Dw^2\cos\vartheta'$, with D as an undetermined coefficient

* We only need to know the ratio of M to D, because it's the balance of mechanical forces and drag forces that are changing
* In still air, the cyclist then exerts a force $F = F_m + F_d$
* His or her power output is force times velocity,

$$P = Fv = Mv^2 + Dw^2\cos\vartheta' v$$

* His or her maximal power output, as measured in still air, is then

$P = Mv_0 + Dv_0^3$, when $u=0$ and $\cos\vartheta' = 1$

* Assume that drag forces predominate, at 10 times the power demand of the mechanical forces

$$Mv_0^2=0.1Dv_0^3$$

* Now assume an actual speed, to get some numerical values. Let's pick $v_0 = 15$ (meters per second), or about 33 mph. This is a strong cyclist on a good bike.

We now have, dividing the last equation by v_0^2,

$$M = 0.1Dv_0 = 1.5D$$

We also have a power output, in some units:

$$P = 1.5D*225+D*3375 = 3712.5\ D$$

This is the power the cyclist will be putting out with any nonzero windspeed in a direction given by the angle θ. He or she will hit a new forward speed, v, which we now have to determine by making the power output equal the baseline value in still air. We can write

$$P = D[1.5v^2 + (v^2 + u^2 + vu\cos\vartheta)\cos\vartheta'] = 3712.5\ D$$

We can obviously cancel out the factor D on both sides. The trick is now, for any specified u and ϑ, to solve this equation. It helps to write an equation that must equal zero (the difference between the estimated power and the achievable power), as

$$g = 3712.5 - [1.5v^2 + (v^2 + u^2 + vu\cos\vartheta)\cos\vartheta'] \rightarrow 0$$

There is no analytical or "closed form" solution, so we'll solve this numerically. I set this up in the latter half of the spreadsheet. I solved it by successive guesses. There's a more automated method, the Newton-Raphson method for finding the root of an equation. For any estimated value of v, evaluate the function g. Also evaluate its derivative with respect to v, $g' = dg/dv$. The next guess should be $v - g/g'$. This works nicely if we're close enough to the correct value that g is nearly linear in v. We won't use this, because the analytic form for dg/dv is so complicated algebraically. We just keep guessing. Here are some results for the case of a strong wind, u = 20 (faster than the cyclist's "free air speed").

Max power (divided by D)=	3712.5				
Set u=	20	m/s			
Set θ=	0.1	degrees	=	0.001746	radians
Guess for v=	5.6	m/s			
cosθ=	0.99999848		sin(theta)	0.001746	
tanθ'=	0.00136368		v+u*cos(theta)	25.59997	
θ'=	0.00136368				
1.5v^2=	47.04				
v*(v^2+u^2+2vu cosθ)cosθ'=	3670.01068				
g=	-4.5506767				

Results	u	θ (degrees)	v	
	10	180	21.75	
		150	20.45	
		120	17.40	
		90	14.07	
Excel gets wrong quadrant for θ<0		60	11.46	
Need an IF statement to fix this		30	9.94	
		0	9.46	
	20	180	29.15	
		150	25.57	
		120	18.76	
		90	12.20	
		60	8.03	
		30	6.12	
		0	5.60	Had to use 0.1 deg. or get wrong quadran

You can see that a wind that's somewhat lower than the cyclist's "free air speed" exerts a penalty of about 1 m s^{-1} out of 15 total when it's a pure crosswind. It adds 45% (6.75 m s^{-1}) to that speed when it's a tailwind and it cuts 36% (4.54 m s^{-1}) when it's a pure headwind. A wind stronger that the cyclist's free air speed exact a bigger penalty as a crosswind (17%, or 2.8 m s^{-1}), and a very large penalty (63%, or 9.4 m s^{-1}) as a headwind. It's a great boost as a tailwind, almost doubling the speed. Note that the cyclist can still make headway against a wind $u > v_0$; this is not true for a vehicle moving in the air, not exerting force on the ground.