How far can one see in a (heavy) rainstorm?

Basic approach: figure out the density of drops per volume and then their obscuration fraction per unit viewing length through the rain, and use Beers' law for the integrated obscuration per unit length.

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1. Intensity of rain and its relation to the rate at which a scene is obscured along a line of sight

Define intensity *I* = depth of rain delivered to a horizontal surface per unit time. This is commonly quoted in mm of water per hour (we'll convert it to SI units of m s^{-1} to use it in calculations).

$$
\begin{aligned}
Intensity &= (velocity\ of\ drops)(volume\ of\ drops\ per\ volume\ of\ air) \\
I &= u * (volume\ per\ drop)(number\ density\ of\ drops) \\
&= u V_{drop} \rho_{drops}
\end{aligned}
$$

What fraction of the area, *A*, is obscured in viewing a distance *dx* through this distribution of drops?

Consider a square viewing area, *A*, through which we look perpendicularly a distance *dx*. This encloses a volume *A dx*. In this volume, we have a number of drops

$$N = A\,dx\,\rho_{drops}$$

Each drop obscures an area equal its cross-sectional area, a_{drop}. For a spherical drop, which a raindrop is not quite, we have $a_{drop} = \pi r^2$, with *r* = the radius of the drop.

The fractional obscuration is $N\ a_{drop}$, if the distance *dx* contains few enough drops that their areas don't overlap (we can always choose *dx* small enough). We then have the fractional obscuration, *f*, as

$$f = A\,dx\,\rho_{drops} a_{drop} / A = \rho_{drops} a_{drop} dx$$

We've assumed that all drops are the same size. We can get more precise, with more complicated math, if we assume a distribution of drops.

The rate at which obscuration fraction, *F*, rises with distance is

$$\frac{dF}{dx} = \rho_{drops} a_{drop}$$

This is the negative of the rate at which visibility declines.

2. Integrating the effect of obscuration over distance

Let's derive the integration over distance a couple of ways.

First, consider "stacking" similar volumes along the line of sight. The first volume leaves a fraction *Vis=(1-f)* as unobscured ("*Vis*" obviously means "visible fraction). Adding another volume of the same length *dx* leaves a fraction $(1-f)(1-f) = (1-f)^2$, if the positions of the drops are not correlated between the two volumes (a very good approximation!). Three volumes in a row gives us an unobscured fraction $(1-f)^3$. Continuing, for a number *n* of such volumes, we have the unobscured fraction as $(1-f)^n$. Let us consider a distance *x* that is covered by n such volumes, or *x = n dx*, and thus *n = x/dx*. We then have

$$Vis = (1 - \rho_{drops} a_{drop}\, dx)^{x/dx} == (1 - k\, dx)^{x/dx}$$

In the limit that *dx* is very small, this simply yields

$$Vis = e^{-kx}$$

where *e* is the base of the natural logarithms. If this is not familiar, you can look up various explanations.

The other way of solving for *Vis* over any finite distance x is to use calculus. We have the relation between visibility at one distance x and visibility at an incremented distance *x+dx* as

$$Vis(x+dx) = Vis(x)(1 - k\, dx)$$

That is, there is a rate of change of visibility with distance, *d(Vis)/dx*, which is

$$\frac{d(Vis)}{dx} = \frac{\lim}{dx \to 0} \frac{Vis(x+dx) - Vis(x)}{dx} = \frac{\lim}{dx \to 0} \frac{-Vis(x)k\, dx}{dx} = -k\, Vis(x)$$

This equation readily integrates to

$$Vis = e^{-kx}$$

as above.

3. Figuring out drop number density in rain: we need intensity, drop size, and the drops' vertical speed

OK, now we have to figure out, for a given drop size, what are the magnitudes of the drop number density, ρ_{drops}, and drop cross-sectional area, a_{drop}.

For this, we have to know the size of the drops, given by their radius *r* as (nearly) spheres. The drop area is simple to compute, as πr^2. The drop number density is trickier. It depends on how fast the drops are falling – that is, what is their vertical speed, *u*? Clearly, the intensity, which we assume that we know, is related to drop speed and other characteristics as

$$I = (number\ density\ of\ drops)(volume\ per\ drop)(velocity\ of\ drops)$$
$$= \rho_{drops} V_{drop}\, u$$

We can readily rearrange this to

$$\rho_{drops} = \frac{I}{V_{drop} u}$$

We should know *I* and *r*, hence also V_{drop}, so we need now to estimate *u*. We assume that the drops have reached their terminal velocity, at which point the gravitational force pulling a drop down equals in magnitude the drag force retarding its flight.

4. Estimating the terminal velocity (speed) of a falling raindrop, from gravitational force = drag

For a spherical drop that has the almost immutable density of water, ρ_w, and radius *r*, its mass is simply its volume multiplied by its density. We multiply the mass by the acceleration of gravity, *g*, to get the gravitation force on the drop, F_g, as

$$F_g = \frac{4}{3}\pi r^3 g$$

To get the drag force, we use the standard formula for drag,

$$F_d = \frac{Energy\ imparted\ to\ air\ displaced)}{distance}$$
$$= \frac{(mass\ of\ air\ displaced)\left(velocity\ squared\right)}{distance}$$

Now, if a sphere, such as a raindrop simply moves aside a volume of air equal to the drop cross-sectional area, *A*, multiplied by the distance it travels in a short time *dt*, then the mass of air moved, *dm*, would be

$$dm = \rho_{air}(Volume\ of\ air)$$
$$= \rho_{air} A u\, dt$$

and the energy imparted would be this mass multiplied by ½ u^2 (energy = ½ mass * square of velocity). We would then get

$$F_d = \frac{1}{2}\frac{\rho_{air} A u\, dt\, u^2}{u\, dt} = \frac{1}{2}\rho_{air} A u^2$$

5. Some details of the drag force

Because the air flow around an object depends upon the object's shape and even the speed of the object, there is a correction factor, the coefficient of drag, making the proper form to be

$$F_d = \frac{1}{2} C_d A \rho_{air} u^2$$

For a sphere, the value of the drag coefficient, C_d, is a known function of its speed, *u*. We'll see shortly that, at the usual range of speeds of raindrops, it's about 0.5. We'll keep the full formula for now, and we'll equate the gravitational force to the drag force:

$$\frac{4}{3}\pi r^3 \rho_w g = \frac{1}{2} C_d A \rho_{air} u^2$$

We can rearrange this to solve for the speed, *u*:

$$u^2 = \frac{8}{3}\frac{\pi r^3 \rho_w g}{C_d \pi r^2 \rho_{air}} = \frac{8}{3}\frac{\rho_w g}{C_d \rho_{air}} r$$

So, the speed of raindrops falling varies roughly with the square root of their radius.

This is really a transcendental equation in *u*, because C_d depends on *u*, with a close approximation being

$$Cd = 0.4 + \frac{24}{\text{Re}} + \frac{6}{1+\sqrt{\text{Re}}}$$

where *Re* is the Reynolds number, a dimensionless quantity that is the ratio of inertial forces on an object to the viscous forces on it. It is defined as

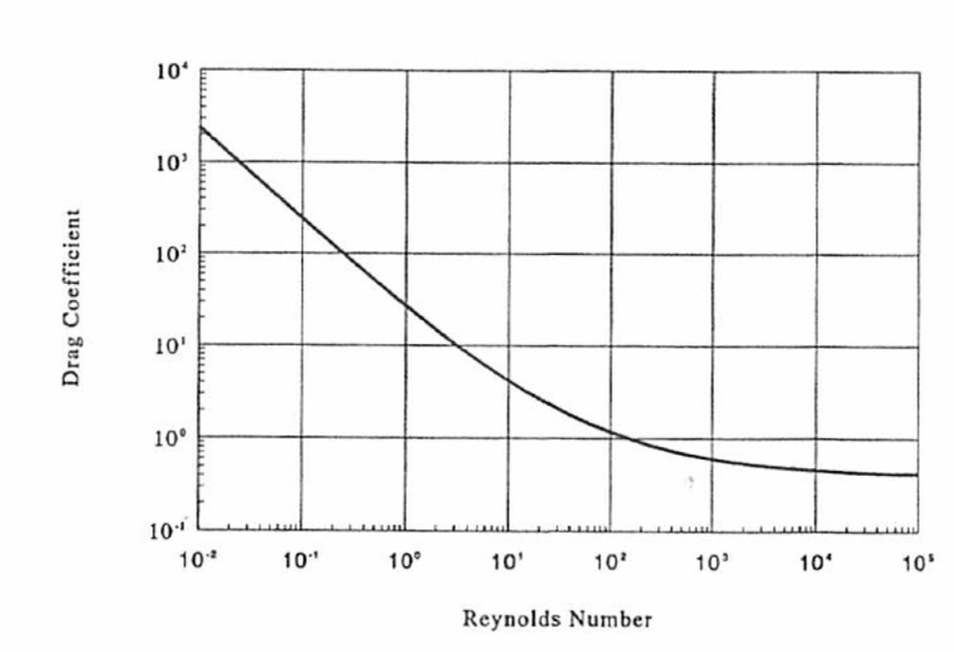
$$\text{Re} = \frac{u\, l_c}{\nu}$$

Here, *u* is the speed we have been considering, l_c is a "characteristic length"over which the fluid (air) is flowing (we take it as the diameter of the sphere), and *v* is the kinematic viscosity of air, a known quantity that depends on air temperature; it's about 1.5×10^{-5} $m^2 s^{-1}$ near 20°.

We could solve the equation relating the gravitational and drag forces for the value of *u* by iteration or other suitable methods, but we can also make a very good approximation that C_d is close to 0.5. Let's guess that *u* is several meters per second, say, 3 m s^{-1}. For a drop diameter of 3 mm (pretty big, as it would be in a heavy rain), we have

$$\text{Re} = \frac{(3 m\, s^{-1})(3x10^{-3}\, m)}{1.5x10^{-5}\, m^2\, s^{-1}} = 600$$

Let's look at a graph taken from p. 115 in a fascinating book, M. W. Denny, *Air and Water: the Biology and Physics of Life's Media* (Princeton University Press, Princeton, NJ, 1994):



We see that C_d is between 0.5 and the limiting value 0.4 in this range; taking it as 0.5 is good enough. (Note: C_d actually declines at much higher Reynolds numbers – see, for example, p. 92 in another great book, S. Vogel, Life in Moving Fluids, Princeton University Press, Princeton, NJ, 1994, but these numbers are beyond what raindrops or biological organisms experience.)

6. Finally, we can compute the falling speed of a raindrop

Whew! Now let's get the numerical values. We have

$$u = \sqrt{\frac{8}{3}\frac{\rho_w g}{C_d \rho_{air}} r}$$

At common conditions, the ratio of the densities of water and air, ρ_w/ρ_{air}, is about 860, and the acceleration of gravity is close to 10 m s^{-2}. With C_d equal to 0.5 and r = 1.5 mm = 1.5x10^{-3} m, we get u = 8.3 m s^{-1}. That's makes for a little wallop when it hits us, as we all know.

Let's use this number to compute the number density of drops in air in this hard rain:

$$\rho_{drops} = \frac{I}{V_{drop} u}$$

We need to know how intense a hard rain is. The hardest storms can hit 150 mm per hour. More in our common experience might be 30 mm per hour, or 30x10^{-3} m/(3.6x10^{3} s) = 8.3x10^{-6} m s^{-1}. The drop volume is the familiar (4/3)πr^3, or 1.41x10^{-8} m^3. With u = 8.3 m s^{-1}, we get

$$\rho_{drops} = \frac{8.3x10^{-6}\, m\, s^{-1}}{(1.41x10^{-8}\, m^3)(8.3 m\, s^{-1})} = 71 m^{-3}$$

So, there are about 71 drops in a cubic meter of air. It feels like a lot more!

7. We arrive at the obscuration coefficient that tells us how fast visibility declines with distance

Finally, the obscuration coefficient, *k*, is

$$k = \rho_{drops} a_{drop} = (71 m^{-3})(3.14 x 2.25 x 10^{-6} m^2)$$
$$= 5.0 x 10^{-4} m^{-1}$$
$$= 0.5 km^{-1}$$

That is, about ½ of the visibility is lost in viewing through 1 km of such heavy rain. To be more accurate, the fractional visibility is $e^{-0.5}$ = 0.61, so that 39% of the visibility is lost. To lose 90% of the visibility, one would have to view through a distance such that

$$e^{0.5x} = 0.1$$

with *x* in km, or

$$x = -2\ln(0.1) = 4.6 km$$

In reality, the obscuration might be worse than computed, because the scattered light degrades the contrast in the scene we are viewing.

8. What about lighter rainfalls (mists)?

What about mists? Let's take a mist falling at 1 mm per hour, with droplets that are only 1 mm in diameter. We can go through all the calculations again, a bit more quickly now. Let's start with calculating the vertical speed of these droplets. They are 1/3 as large as the big raindrops, so their terminal speed is $\sqrt{1/3}$ as fast, or 4.8 m s^{-1}. We can compute ρ_{drops} as

$$\rho_{drops} = \frac{10^{-3} m}{3.6 x 10^3 s} / [(\frac{4}{3}\pi \ 0.125 x 10^{-9} m^3)(4.8 m s^{-1})]$$
$$= 111 m^{-3}$$

We then have *k* = (111 m^{-3})(3.14x0.25x10^{-6} m^2) = 8.7x10^{-5} m^{-1} = 0.087 km^{-1}. One can see a long distance in such a mist.

9. What about fogs?

What about fog that basically hangs in the air, as very fine droplets? This is really good at obscuring vision. The area per drop varies as r^2 but the number density of drops varies as $1/r^3$. Thus, for a given water content per volume of air, the extinction coefficient, *k*, as the product of drop area and drop number density, varies as *1/r*; small drops are more effective. We'd have to know the mass or volume of water per unit volume of air in fogs, and the droplet sizes. An area-weighted mean diameter might be 5 micrometers (see, for example, R. G. Eldridge. 1961. A few fog drop-size distributions. Journal of Meteorology 18:671-676) and the number density might be about 300 per cubic centimeter, or about 3x10^8 per cubic meter! The magnitude of a_{drop} is then 3.14x 6.25x10^{-12} m^2, and the value of *k* is about

$$k = (3 x 10^8 m^{-3})(3.14 x 6.25 x 10^{-12} m^2) = 0.006 m^{-1} = 6 km^{-1}$$

Then, the distance one can see 10% of the scene is

$$x = \ln(0.1)/6 = 0.77\, km$$

Some fogs are much more “potent” than this.

Note what the volume fraction of water in air is for the example cited. The volume per drop, V_{drop}, is $(4/3)\pi(2.5\text{x}10^{-6}\text{ m})^3 = 6.5\text{x}10^{-17}\text{m}^3$. At a number density of $3\text{x}10^8$ per cubic meter, the volume of water per unit volume of air is only $(3\text{x}10^8\text{ m}^{-3})(6.5\text{x}10^{-17}\text{ m}^3) = 2.0\text{x}10^{-8}$. It’s possible for fogs to hit 10^{-6} in volume fraction, or 50 times higher than in the example. This would make the 10% visibility distance 1/50 as large, or about 15 meters!

A note

P.S.: Another very interesting book for math in nature and biology is John A. Adam, A Mathematical Nature Walk (Princeton University Press, Princeton, NJ, 2009).