Adiabatic lapse rate in the atmosphere

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A parcel of air, moved uphill (or just up in the air), expands, and therefore does work against the surroundings. This extracts energy. With no source of new energy (poor transfer of heat across large air masses), this means that the internal energy must drop - that is, the temperature must drop.

We can calculate the rate of T drop with elevation, the "adiabatic lapse rate."

We use another principle of physics, that the pressure difference from top to bottom of an air parcel must be big enough to support the air mass from sinking in the gravitational field. This sets the profile of pressure vs. height.

Derivation:

Hydrostatic equilibrium says how P varies with height (and density)

Consider a parcel of air of area A; base is at height y, top at height y+dy

Force on top is (P+dP)A, on bottom is P A; difference is A dP, driving parcel of air up (dP is negative; P decreases with height)

Gravitational force on the parcel is -mg, where mass $m = \rho M_w V$

with ρ = molar density, M_w = mass per mole (molecular wt.) and V=volume=A dy

Force balance: $A\, dP = -\rho M_w A\, dy\, g$

or $dP/dy = -\rho M_w g$

Now express density ρ in terms of pressure.

First, use ideal-gas law to express ρ in terms of P and T:

$PV=nRT$; $\rho = n/V = P/(RT)$

How do we relate P and T uniquely? If air is displaced, it settles back in place (it's in equilibrium at all heights)...and the displacement is an adiabatic process - no heat is added or subtracted from air parcel.

$dQ = 0 = dU - dW$ (change in energy content = 0 = change in internal energy - change in work done on surroundings - just conservation of energy)

That is, $dU = dW$ (change in internal energy = loss from work done).

Now express these two changes in terms of changes in pressure and temperature:

$C_v\, dT = -P\, dV = -(RT/V)\, dV$

Take all the T's on one side, all the V's on the other side:

$C_v\, dT/T = -R\, dV/V$

Integrate it from initial state to final state:

$C_v \ln(T/T_0) = -R \ln(V/V_0)$

$\ln(T/T_0) = -(R/C_v) \ln(V/V_0)$

For a gas of diatomic molecules that can move (translate) and rotate freely, we have C_v = (5/2) R (each "degree of freedom" of motion has a heat capacity of (1/2) R, and there are 5, two rotations and three translational directions).

Exponentiate both sides:

$T/T_0 = (V/V_0)^{-2/5}$, or

$V/V_0 = (T/T_0)^{5/2}$

Now use this to express P variations in terms of T variations:

$P/P_0 = (RT/V) / (RT_0/V_0) = T V_0/(T_0 V) = (T/T_0) (T/T_0)^{5/2} = (T/T_0)^{7/2}$

Invert this, to express T changes in terms of P changes:

$T = T_0 (P/P_0)^{2/7}$

Use this in the density equation

$P/(RT) = P / [RT_0 (P/P_0)^{2/7}$ --> $P^{5/7} P_0^{2/7}/(RT_0)$

Finally, let's integrate the equation of hydrostatic equilibrium. Recall that this was

$dP/dy = \rho M_w g$, and use $\rho = P/(RT)$:

$dP/dy = -P^{5/7} [P_0^{2/7} M_w g]/[RT_0] == -k P^{5/7}$ --> $dP/P^{5/7} = -k dy$

Integrating from the ground (y=0) to any height y:

$\int dP P^{-5/7} = (P^{2/7}-P_0^{2/7})/(2/7) = -k y$

$P^{2/7}-P_0^{2/7} = -(2/7) [P_0^{2/7} M_w g / (RT_0)] y$

$(P/P_0)^{2/7} -1 = -(2/7)P_0^{-2/7}[P_0^{2/7}g M_w /(RT_0)] y = -(2/7) [g M_w/(RT_0)] y$

$P/P_0 = [1 - (2/7) [g M_w/(RT_0)] y]^{7/2} == [1 - (2/7)ay]^{7/2}$

At small y (near the surface), $[1 - (2/7) a y]^{7/2} \approx 1 - (7/2)(2/7)ay) = 1- ay$

- that is, P falls off linearly with height- about 1% per 80 meters, or e-fold (to 37% of sea-level pressure) in one "scale height" of 8000 m (near top of Mt. Everest), or about 15% at elevation of Las Cruces (1200 m)

Now let's convert from the P profile to the T profile:

$T/T_0 = (P/P_0)^{2/7} = [1-(2/7) ky]^{(7/2)(2/7)} = 1 - (2/7) ky$!

or,

$T = T_0 - (2/7) [g M_w/R] y = T_0 - b y$

The factor in front of y can be evaluated, using $g = 9.8\ m\ s^{-2}$, $M_w = 0.029\ kg\ mol^{-1}$ and R = $8.314\ J\ mol^{-1}\ K^{-1}$, to give $b = -0.0098\ K\ m^{-1}$,

that is, 9.8 degrees (Kelvin or Celsius) per 1000 m.

Surprising result: temperature drops linearly with elevation. (It could drop toward absolute zero at 28 km above sea level, except that the absorption of sunlight becomes important as an energy source in the upper atmosphere.

Water vapor absorbs near infrared [not important at high elevations; there's not much water]; CO_2 absorbs thermal infrared everywhere; and ozone absorbs UV, esp. in the stratosphere. Above the stratosphere, the air even gets warmer with elevation.)

Wet adiabatic lapse rate:

We have to consider how condensation of water vapor (as the air cools) releases sensible heat. This slows the rate of temperature decrease with height. There is no analytical solution, but numerically we can find that the rate is about 4 degrees per 1000 m at low elevations, increasing slowly and steadily with elevation to the dry rate of 9.8 degrees per 1000 m as water vapor becomes scarce much faster than the pressure drops.

Often, people cite a mean rate of about 6 or 6.5 degrees per 1000 m, but this is a rough approximation.

Consequences:

Colder life zones with rising elevation/ "mirroring" the N-S gradient

Other implications: water vapor condenses out fast with height

-> clouds form as air lifts

-> total water content of air is very limited; equivalent to 2.5 cm depth of liquid water, over the globe. -> water turns over rapidly in the atmosphere (every 9 days)

-> many interesting pheonomena: we can track water loss / sources of rain by the T at which they condensed (the isotopic composition is indicative)