A corrected and improved discussion of sling psychrometry: relations among T_{wet}, T_{air}, T_{dew}, P_{air}, and humidity

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The method of sling psychrometry is to make the boundary layer conductance at the two thermometer bulbs to be so large that only two terms matter in the energy balance: evaporative cooling, Q_E, and convective heat transfer, Q_C. At steady state, the sum of these two terms is zero, which we may write as

$$0 = -\frac{\lambda g_b}{P}\,[e_s(T_{wet}) - e_a] - g_b C_P (T_{wet} - T_{air}) \tag{1}$$

Here I use the notation $e_s(T_{wet})$ for saturated vapor pressure as a function of T_{wet}. I also set up the notations here:

λ = molar heat of vaporization of water (about 44,000 J mol^{-1})
g_b = boundary-layer conductance, in molar units, as commonly used in plant physiology (mol m^{-2}s^{-1})
P = total air pressure
$e_s(T_{wet})$ = saturated vapor pressure of water (Pa) at the wet-bulb temperature
T_{wet} = wet-bulb temperature
e_a = partial pressure of water vapor in ambient air
C_P = molar heat capacity of air (about 29 J mol^{-1})

We can factor out g_b and rearrange Eq. (1) to

$$T_{air} - T_{wet} = \frac{\lambda}{PC_P}\,[e_s(T_{wet}) - e_a] \tag{2}$$

For a first approximation of how the wet-bulb temperature is related to the environmental conditions, we can express $e_s(T_{wet})$ using a linear variation of saturating vapor pressure with temperature:

$$e_s(T_{wet}) = e_s(T_{air}) + e' \cdot [T_{wet} - T_{air}] \tag{3}$$

This uses the fact that the value of any smoothly-varying function, f(x), at a new point, x',can be approximated as its value at any the original point, x, plus its derivative, df/dx, multiplied by the offset, x'-x. In the current case, e' is clearly the derivative, We can use a quadratic form for higher accuracy, but the principle remains the same.

The numerical value of the derivative, $e' = de_s(T)/dT$, can be looked up in a table. It can also be calculated from a highly accurate mathematical formula from the Smithsonian Tables,

$$e_s(T) = 610.8 \text{ Pa} \exp[\frac{17.269\ T}{237.2 + T}] \tag{4}$$

The analytical form of the derivative, derived using some calculus and algebra, is simply

$$e'(T) = e_s(T)\,\frac{4098}{(T + 237.2)^2} \tag{5}$$

These equations work very well for temperatures above freezing. There are similar forms for temperatures below freezing. For T = 20 (all T values here are in degrees Celsius), we can see that e' = 0.062 $e_s(T)$ - that is, vapor pressure rises about 6.2% per degree Celsius rise in temperature. To complete the calculation at this temperature, we can plug T = 20 into Eq. (4) to get $e_s(20)$ = 2339 Pa and e' = 145 Pa per K (K = Kelvin, same scale as Celsius). We can do the same thing for an example, T = 30, that we'll use later, below, to get $e_s(30)$ = 4246 Pa and e' = 244 Pa/K.

Substituting Eq. (3) into Eq. (2), we obtain

$$T_{air} - T_{wet} = \frac{\lambda}{PC_P}\,[e_s(T_{air}) - e_a - e' \cdot (T_{air} - T_{wet})] \tag{6}$$

We can write e_a as the the saturated vapor pressure at ambient temperature, multiplied by the relative humidity as a fraction, h_r. This makes the first two terms in brackets on the right become $e_s(T_{air}) \cdot (1 - h_r)$. Then we can bring all the terms in temperature to the left-hand side, to obtain

$$[T_{air} - T_{wet}][1 + \frac{\lambda e'}{PC_P}] = \frac{\lambda}{PC_P}\,e_s(T_{air})\,[1 - h_r] \tag{7}$$

The factor on the right-hand side after $\lambda/(PC_P)$ is just the vapor-pressure deficit, D. I write relative humidity, h_r, as a fraction (e.g., 40% = 0.4, literally). Clearly, we can solve for the temperature difference as

$$T_{air} - T_{wet} = \frac{\lambda D}{PC_P[1 + \frac{\lambda e'}{PC_P}]} \tag{8}$$

Here is a numerical example:

$T_{air} = 30°C \rightarrow e_s(T_{air}) = 4246$ Pa, e' = 244 Pa K^{-1}
$h_r = 0.5 \rightarrow e_a = 2123$ Pa
$P = 10^5$ Pa
λ, C_P as above.

Substituting these numerical values into Eq. (5) yields $T_{air} - T_{wet} = 6.85°C$. [Note: this is slightly different from a presentation made in 2001, in which the value of e' was not done as accurately.]

This value is not quite accurate, since we used a linear approximation for the change in $e_s(T)$ with a change in temperature. The vapor pressure behaves as the more upwardly curved exponential form in Eq. (4). Of course, we can't get a simple (that is, analytical) solution to Eq. (2), which is then a transcendental (nonpolynomial) equation. We can, however, iterate the solution:

Let's call the solution we got above the lowest approximation to the true value of T_{wet}, or T^0_{wet}. The next and better solution we'll call T^1_{wet} and write it as T^0_{wet} + dT, where dT is a correction that should be small. We insert this into Eq. (2) and get

$$T_{air} - T_{wet} - dT = \frac{\lambda}{PC_P}[e_s(T^0_{wet} + dT) - e_a] \tag{9}$$

$$= \frac{\lambda}{PC_P}[e_s(T^0_{wet}) + e'(T^0_{wet}) \cdot dT - e_a]$$

Here we have used the linear approximation again, but it should be better, because we are closer to the true value of T_{wet}, and all smooth functions look linear over short ranges.

We can gather all the terms in dT on one side, to get

$$T_{air} - T^0_{wet} - \frac{\lambda}{PC_P}[e_s(T^0_{wet}) - e_a] = dT \cdot [1 + \frac{\lambda e'(T^0_{wet})}{PC_P}] \tag{10}$$

It's easy to solve this for dT. Using the values from the first approximation, we get dT = -1.10, so that our new estimate for T_{wet} is 23.15 - 1.10 = 22.05°C. Doing one more round of approximation gives us a further correction of -0.03°C, or an estimate of T_{wet} as 22.02°C. We can check that this is good by plugging it into Eq. (2) again, getting

$$30 - 22.02 = \frac{44,000}{29x10^5}[2648 - e_a] \tag{11}$$

The factor in brackets then has to be 526 Pa, and e_a is shown to be 2648 - 526 = 2122 Pa, very close to the exact value of 2123 Pa.

The real value of T_{wet} in our test case is then 22.02°C. We would compute the proper value of e_a, if we used the accurate formula in Eq. (4) for $e_s(T_{wet})$, and we would get the proper value of relative humidity, 0.5 or 50%.

If we were interested in the dew-point temperature, T_{dew}, we could compute it (or make a table to do this), using the relation that the air at this temperature has a saturated vapor pressure equal to e_a:

$$e_s(T_{dew}) = h_r e_s(T_{air}) = e_a \tag{12}$$

While the saturated vapor pressure has that seemingly difficult exponential form, we can actually solve for T_{dew} readily:

$$610.8 \exp[\frac{17.269\ T_{dew}}{(237.2 + T_{dew})} = \frac{e_a}{610.8} \tag{13}$$

Taking logarithms of both sides, we get

$$\frac{17.269\ T_{dew}}{(237.2 + T_{dew})} = \ln(e_a/610.8) \tag{14}$$

Multiply through by the denominator on the left, 237.2 + T_{dew}, and call the right-hand side simply L, to get

$$17.269\ T_{dew} = (237.2 + T_{dew})\ L \tag{15}$$

or

$$[17.269 - L]\, T_{dew} = 237.2\, L \tag{16}$$

In the case above, we readily obtain L = 1.246 and $[17.269 - 1.246]T_{dew} = 237.2*1.246$, or T_{dew} = 18.45°C. This is below the wet-bulb temperature, as it must be.

I'm sure there are derivations of these relations in texts, but none that I can find at hand. The principles are the same in any derivation.

There are some other approximations here, such as assuming that the heat of vaporization of water is constant, independent of temperature. By the laws of thermodynamics, it does vary slightly. The change in our calculation if we account for this is small - in fact, smaller than the real offset of wet-bulb temperature that arises because twirling the psychrometer does not give us an infinite conductance in the boundary layer. If we really want extreme accuracy in measuring humidity or dew-point, there are other technologies (all of them more expensive, of course).